The Quantum-Classical and Mind-Brain Linkages: The Quantum Zeno Effect in Binocular Rivalry

Henry P. Stapp

Theoretical Physics Group

Lawrence Berkeley National Laboratory

University of California

Berkeley, California 94720

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Abstract

A quantum mechanical theory of the relationship between perceptions and brain dynamics based on von Neumann's theory of measurements is applied to a recent quantum theoretical treatment of binocular rivalry that makes essential use of the quantum Zeno effect to give good fits to the complex available empirical data. The often-made claim that decoherence effects in the warm, wet, noisy brain must eliminate quantum effects at the macroscopic scale pertaining to perceptions is examined, and it is argued, on the basis of fundamental principles, that the usual decoherence effects will not upset the quantum Zeno effect that is being exploited in the cited work.

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I. INTRODUCTION

Efstratios Manousakis[1] has recently given a quantum mechanical description of the phenomena of binocular rivalry that fits the complex empirical data very well. It rests heavily upon the quantum Zeno effect, which is a strictly quantum mechanical effect that has elsewhere[7–11] been proposed as the key feature that permits the free choices on the part of an observer to influence his or her bodily behavior. The intervention by the observer into physical brain dynamics is an essential feature of orthodox (Copenhagen/von Neumann) quantum mechanics. Within the von Neumann quantum dynamical framework this intervention can, with the aid of quantum Zeno effect, cause a person's brain to behave in a way that causes the body to act in accord with the person's conscious intent. On the other hand, using a quantum mechanical effect in brain dynamics might seem problematic, because it depends on the existence of a true macroscopic quantum effect in a warm, wet, noisy, brain. It has been argued that some such effects will be destroyed by environmental decoherence[4]. Those often cited arguments cover certain macroscopic quantum mechanical effects, but they fail, for the fundamental reasons described below, to upset the quantum Zeno effect at work here.

II. COUPLED OSCILLATORS IN CLASSICAL PHYSICS

It is becoming increasingly clear that at least some of our normal conscious experiences are associated with $\sim 40Hz$ synchronous oscillations of the electromagnetic fields at a collection of brain sites[2, 3]. These sites are evidently dynamically coupled. And the brain appears to be, in some sense, approximately described by classical physics. So I begin by recalling some elementary facts about coupled classical simple harmonic oscillators (SHOs).

In suitable units the Hamiltonian for two SHOs of the same frequency is

$$H_0 = \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2). \tag{1}$$

If we introduce new variables via the canonical transformation

$$P_1 = \frac{1}{\sqrt{2}}(p_1 + q_2) \tag{2}$$

$$Q_1 = \frac{1}{\sqrt{2}}(q_1 - p_2) \tag{3}$$

$$P_2 = \frac{1}{\sqrt{2}}(p_2 + q_1) \tag{4}$$

$$Q_2 = \frac{1}{\sqrt{2}}(q_2 - p_1), \tag{5}$$

and replace the above H_0 by

$$H = (1+e)(P_1^2 + Q_1^2)/2 + (1-e)(P_2^2 + Q_2^2)/2, \tag{6}$$

then this H expressed in the original variables is

$$H = H_0 + e(p_1 q_2 - q_1 p_2). \tag{7}$$

If $e << 1$ then the term proportional to e acts as a weak coupling between the two SHOs whose motions for $e = 0$ would be specified by H_0.

The Poisson bracket (classical) equations of motion for the coupled system are, for any x,

$$dx/dt = \{x, H\} = \sum_j \Big(\frac{\partial x}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial x}{\partial p_j}\frac{\partial H}{\partial q_j}\Big). \tag{8}$$

They give

$$dp_1/dt = -q_1 + p_2 e \tag{9}$$

$$dp_2/dt = -q_2 - p_1 e \tag{10}$$

$$dq_1/dt = p_1 + q_2 e \tag{11}$$

$$dq_2/dt = p_2 - q_1 e. \tag{12}$$

A solution is

$$p_1 = \frac{C}{2}[\cos(1+e)t + \cos(1-e)t] \tag{13}$$

$$q_2 = \frac{C}{2}[\cos(1+e)t - \cos(1-e)t] \tag{14}$$

$$p_2 = \frac{C}{2}[\sin(1+e)t - \sin(1-e)t] \tag{15}$$

$$q_1 = \frac{C}{2}[-\sin(1+e)t - \sin(1-e)t]. \tag{16}$$

A common phase ϕ can be added to the argument of every sine and cosine.

These equations specify the evolving state of the two SHO system by a trajectory in (p_1, q_1, p_2, q_2) space.

When we introduce the quantum corrections by quantizing this classical model we obtain an almost identical quantum mechanical description of the dynamics. In the very well known way the Hamiltonian H_0 goes over to (I use units where Planck's constant is 2π.)

$$H_0 \;=\; \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2) = (a_1^\dagger a_1 + 1/2) + (a_2^\dagger a_2 + 1/2). \tag{17}$$

The connection between the classical and quantum descriptions of the state of the system is very simple: the point in (p_1, q_1, p_2, q_2) space that represents the classical state of the whole system is replaced by a "wave packet" that, insofar as the interventions associated with observations can be neglected, is a smeared out (Gaussian) structure centered for all times exactly on the point that specifies the classical state of the system. That is, the quantum mechanical representation of the state specifies a probability distribution of the form $(exp(-d^2)\ )$ where d is the distance from a center (of-the-wave-packet) point $(p_1, q_1, p_2, q_2\ )$, which is, at all times, exactly the point (p_1, q_1, p_2, q_2) that is the classical representation of the state.

According to quantum theory, the operator $a_i^\dagger a_i = N_i$ is the number operator that gives the number of quanta of type i in the state.

Thus in the absence of any observations the classical and quantum descriptions are almost identical: there is, in the quantum treatment, merely a small smearing-out in (p, q)-space, which is needed to satisfy the uncertainty principle.

This correspondence persists when the coupling is included. The coupling term in the Hamiltonian is

$$\begin{aligned} H_1 \;&=\; e(p_1 q_2 - q_1 p_2 - p_2 q_1 + q_2 p_1)/2 \\ &=\; ie/2(a_1^\dagger a_2 - a_1 a_2^\dagger - a_2^\dagger a_1 + a_2 a_1^\dagger). \end{aligned} \tag{18}$$

The Heisenberg (commutator) equations of motion generated by the quadratic Hamiltonian $H = H_0 + H_1$ gives the same equations as before, but now with operators in place of numbers. Consequently, the centers of the wave packets will follow the classical trajectories also in the $e > 0$ case. The radius of the orbit is the square root of twice the energy, measured in the units defined by the quanta of energy associated with frequency of the SHO.

III. APPLICATION

With these well known results in hand, we can turn to the implied physics. The above mathematics shows, for SHOs, a near identity between the classical and quantum treatments, insofar as there are no observations. But if at $t = 0$ an observation occurs that informs the agent/observer that the state is now, say, the state in which the first SHO is active but the second oscillalator is not, so that $p_2 = q_2 = 0$, then equations (9) and (11) show that to first order in time the trajectory of the first particle will be the same as the unperturbed ($e = 0$) trajectory.

The essential point here is that quantum theory, in the von Neumann/Heisenberg formulation, describes the dynamical connection between conscious observations and brain dynamics. [Von Neumann[5] brought the mind-brain connection into the formulation in a clear way, as an application of the orthodox quantum precept that each increment in our classically describable knowledge is represented in the mathamatical language of quantum mechanics by the action of projection operators on the prior state. Heisenberg[6] emphasized that if one wants to understand what is really happening then the quantum state should be regarded as a "potentia" (objective tendency) for a real psycho-physical event to occur.] In order to apply this theory, the classically described brain must first be converted to its quantum form. By virtue of the relationships described in section II, this conversion is particularly easy in case the classical state that is connected to consciousness is a SHO state.

The observer, in order to get information about what is going on about him into his stream of consciousness, must initiate probing actions. According to the elaboration of the theory of von Neumann[5] described in Refs[7–11], the brain does most of the work. It creates, in an essentially mechanical way based on trial and error learning, and also upon the current quantum state of the brain, a query/question. Each possible query is associated with a psychological projection into the future that specifies the brain's computed "expectation" about what the feedback from the query will be. The physical manifestation of this query is called "process 1" by von Neumann. It is a key and necessary element in the dynamics: it resolves ambiguities that are not resolved by the physically described laws of quantum mechanics, and it ties the physical description expressed in terms of the quantum mathematics to our communicable descriptions of our perceptions. This process 1 probing action is *not* the famous statistical element in quantum theory! It is needed both in order to specify what the statistical predictions will be about, and also to tie the quantum mathematics to empirical data, and hence to science.

In order to bring out the essential point, and also to tie the discussion comfortably into the common understandings of neuroscientists, who are accustomed to thinking that the brain is well described in terms of the concepts of classical physics, I shall consider first an approximation in which the brain is well described by classical ideas. Thus the two SHO states that we are focusing on are considered to be aspects of possible states of a classically described brain, which is also providing the potential wells in which these two SHOs move. It is the degrees of freedom of the brain associated with these two SHOs that are, in the simple model being considered here, the possible brain correlates of the consciousness of the observer during the period of the experiment. Hence it is they that are affected by von Neumann's process 1. The remaining degrees of freedom are treated in this approximation as providing the background classically described potential wells in which these consciousness-related SHOs move.

In binocular rivalry experiments two different conflicting images are fed to the two eyes of the observer, and the overall phenomenal result is that generally the perception is of only one scene or the other, with an alternation between the two scenes. Each scene is perceived typically for two to three seconds, but this duration can often be increased if the observer makea a conscious effort to attend more intently to the currently perceived scene.

In our model one of these SHOs corresponds to the brain correlate of the percept associated with one eye, the other SHO is the brain correlate of the percept associated with the other eye. This model reduces the effects of observations, as specified by von Neumann's quantum theory of measurement/observation, to an exactly solvable mathematical problem not clouded by the infinity of effects whose consideration often places any clear rational understanding of the connection between mind and brain beyond a precise intellectual grasp.

In the binocular rivalry context, let the two unperturbed ($e = 0$) motions represent the computed (expected-by-the-brain) evolutions of the brain in the two alternative possible cases when only one eye or only the other is open (or both eyes view the same alternative scene) and let the $e > 0$ case represent the dynamics of the full system in the binocular rivalry case, where the boundary conditions represent a dynamical situation that would allow an SHO component of the electromagnetic field in two disjoint (non-overlapping) regions of the brain to be activated by the two different inputs, so that insofar as no perception occurs both SHO activations could be simultaneously present.

Our experience (apart from the stereoscopic 3-d aspect) is not markedly different when a single scene is being viewed by both eyes or by one, and high-level brain activities associated with these three cases, right-eye, left eye, or both, should not differ greatly. The computed "expectation" naturally follows the unperturbed orbit, which corresponds to normal experience, in which both eyes view essentially the same scene. If $t = 0$ represents the time of the last observation, which projects onto an unperturbed possible state, in which *one* neural correlate is excited, but not the other, so that, say, $p_2 = q_2 = 0$. then (9) and (11) show that for small $t > 0$ the actual brain trajectory in the rivalry $e > 0$ case will diverge only quadratically in time from the $e = 0$ trajectory. This is confirmed by equations (13 - 16).

I assume that there is a rapid sequence of queries at a sequence of times $\{t_1, t_2, t_3, ...\}$. These queries will be based on expectations constructed by the brain on the basis of past experiences. These queries are represented in the quantum mathematics by a series of projection operators $\{P(t_1), P(t_2), P(t_3), ...\}$ [A projection operator P satisfies $PP = P$] This sequence of projection operators represents a sequence of questions that ask whether the current state is on the "expected" track. This track is specified by the $e = 0$ trajectory, which represents expectations based on normal past experience.

To coordinate this physical situation to equations (13-16) we allow the phase ϕ mentioned below these equations to be set at the sequence of process 1 probing times t_i, and let $t > 0$ represent time from the last observation. Note that at each observation time t_i the variable time (t) is set to zero, which makes $p_2 = q_2 = 0$, as required, whereas the p_1 and q_1 will lie (by virtue of the changing ϕ) on the unperturbed ($e = 0$) trajectory. The set of queries follow the unperturbed trajectory.

Up until now I have taken the projections to be projections onto a single quasi-classical state. A projection upon such a state would involve fantastic precision. Each such state is effectively confined to a disc of unit size relative to an orbit radius C of about 10^6 in the units employed in equation (1). [This number 10^6 is the thermodynamic energy per degree of freedom at body temperature, in energy units associated with equation (1), in which Planck's constant is 2π, and the angular velocity is one radian per unit of time. The unit of time in these units is about 4ms for 40 Hertz oscillations.] However, it it is possible (for our SHO case) to define more general operators that are projection operators (i.e., satisfy

$PP = P$) apart from corrections of order, say, $< 10^{-3}$, by using the von Neumann lattice theorem.[12]

If one represents by $[P, Q]$ the projection operator that projects onto the Gaussian state centered at $(p, q) = (P, Q)$, then the lattice theorem says that the following identity holds:

$$\sum [mf, nf] = I \tag{19}$$

where $f = (2\pi)^{1/2}$, where I is the identity operator, and where the sum is over all integer values of m and n except $m = n = 0$. Moreover, the decomposition into different Gaussian components effected by this identity is unique. If one restricts the sum to the lattice points in a very large square region in (p_1, q_1) space then the resulting operator P' is very nearly a projection operator.

For example, if the square region $S(C, 0)$ is centered at the SHO point $(C, 0)$ in the (p_1, q_1) space that we have been discussing, and has sides of length, say, one percent of the radius C of the unperturbed orbit, then each side of the square will be $10^4 f^{-1}$ units compared to the unit size associated with the Gaussian fall off, $\exp(-d^2)$. In this case the associated quasi-projection operator $P' = P(C, 0)$ is essentially a projection operator onto the square region $S(Q, 0)$ of (p_1, q_1) space.

Let $S(C\cos\phi, -C\sin\phi)$ be the square, centered on $(C\cos\phi, -C\sin\phi)$, obtained by rotating $S(C, 0)$ by $-\phi$, so that the line from its center point to the origin is parallel to two of its sides. The action of the unperturbed ($e = 0$) Hamiltonian will take $S(C, 0)$ to $S(C\cos\phi, -C\sin\phi)$ in time ϕ. It will also take $P(C, 0)$ to the quasi-projection operator $P(C\cos\phi, -C\sin\phi)$ associated with the square $S(C\cos\phi, -C\sin\phi)$. These results follow from the simple SHO dynamics in the unperturbed (decoupled) $e = 0$ case.

The classical operation that reports the area of the *intersection* of two regions in classical (p, q) space [which is called "phase space"] goes over in quantum mechanics to the $Trace$ operation acting on the product of the projection operators corresponding to these two regions.

The rules of quantum dynamics are compactly stated in terms of the $Trace$ opration. [The $Trace$ operation acting upon operators/matrices is defined by letting the matrix (or operator) multiplication operation occurring in, say, $Trace AB$ be extended cyclically, so that B acting to the right acts back on A. This means that for any pair of matrices/operators A and B, $Trace\ AB = Trace\ BA$. This property entails also that $Trace\ ABC = Trace\ BCA$. For any X, $Trace\ X$ is a number. In our case, $Trace\ P(P,Q)$ is essentially the area of the square S(P, Q), measured in units of action given by Planck's constant, and $Trace\ P(P,Q)P(P',Q')$ is the area of the intersection of $S(P,Q)$ and $S(P',Q')$.]

The $Trace$ of the product of the "projection" operator $P(p_1(t,e),q_1(t,e))$ centered on the perturbed orbit [where the two arguments are defined by equations (13) and (16)] with the "projection" operator $P(p_1(t,0),q_1(t,0)) = P(C\cos t, -C\sin t)$ centered on the unperturbed orbit is, to lowest order in t, $1-\frac{1}{2}((et)^2 100)$, where for a 40 Hertz SHO the time unit is about 4 ms. The term $\frac{1}{2}((et)^2 100)$ is the ratio of the displacement [of the perturbed square relative to the unperturbed square, namely $\frac{C}{2}(et)^2$], to the length of the side of the square, which is one percent of the radius C of the unperturbed orbit. The unperturbed square rotates rigidly with angular velocity unity, under the action of the unperturbed Hamiltonian, and the lowest-order $e > 0$ displacement is toward the origin $(p_1, q_1) = (0, 0)$. Consequently, the dynamics is essentially unchanged by rotations: the initial condition $(C, 0)$ plays no essential role.

According to the basic precepts of quantum theory, the (physical) "state" of the system at time t is specified by a "density matrix" (or "density operator"), usually denoted by $\rho(t)$. If the answer is 'Yes', then the state just *after* the probing action at time t is $\rho(t+) = P(t)\rho(t-)P(t)$, where $\rho(t-)$ is the state just *before* the time t at which the question is posed. [For answer 'No', $P(t)$ is replaced by its complement $(1 - P(t))$.]

The basic statistical law of quantum theory asserts that, *given the query specified by the projection operator* $P(t)$, the probability that the answer will be 'Yes' is $Trace\ \rho(t+)$ divided by $Trace\ \rho(t-)$.

Note that the query, specified by $P(t)$ and the time t at which $P(t)$ acts, must be specitied *before* the statistical postulate can be applied!

If $\rho(t-)$ is, for the first probing time $t = t_1$, slowly varying over the square domain in (p_1, q_1) space, in the sense that $Trace\ [P,Q]\rho(t-)$ is essentially constant as (P,Q) varies over the square $S(C,0)$, then the state after the initial observation will be essentially the projection operator $P(C,0)$ associated with that initial process 1 probing action. Then our equations show that for any (large) time T the density matrix $\rho(T)$ will be nearly equal to $P(C\cos T, -C\sin T)$ provided the interval T is divided by observations into N equal intervals $t_{i+1} - t_i$, and $N(10eT)^2 N^{-2} << 1$. This condition entails both that *all* the answers will be 'Yes' with probability close to unity, and also that the final $\rho(T)$ will be almost the same as the unperturbed "projection" operator $P(C\cos T, -C\sin T)$.

Thus the rapid sequence of probing actions effectively holds the sequence of outcomes to the *expected* sequence. The affected brain states are constrained to follow the expected trajectory! This is the quantum Zeno effect, in this context.

This result means that if the probing actions come repetitiously at sufficiently short time intervals then the probability that the state will remain on the unperturbed orbit for, say, a full second will remain high even though the perturbed $e > 0$ classical trajectory moves away from the unperturbed orbit by an amount of order C in time T of order e^{-1}.

The drastic slowing of the divergence of the actual orbit from the computed/expected (circular-in-this-case) orbit is a manifestation of the quantum Zeno effect. The representation in the physically described brain of the probing action corresponding to the query "Is the brain correlate of the occurring percept the computed/expected state" is von Neumann's famous process 1, which lies at the mathematical core of von Neumann's quantum theory of the relationship between perception and brain dynamics.

IV. CONCLUSIONS

The bottom line is that, according to this dynamical model, if a person can, by mental effort, *sufficiently increase the rate at which his probing actions occur*, then that person can, by mental effort, quantum dynamically *cause* his brain/body to behave in a way that follows a pre-programmed trajectory, specified, say, by "expectations", instead of following the trajectory that it would follow if the von Neumann process 1 probing actions do not occur in rapid succession. Because the causal origin of the process 1 probing actions *is not specified, even statistically, by the presently known laws of physics,* we have a causal gap that provides a rational place for the experiential aspects of our description of nature to enter, irreducibly, into the theoretical determination of the course of physically described events.

I have focused here on the leading powers in t, in order to emphasize, and exhibit in a relatively simple way, the origin of the key result, which is that for small t on the scale, not of the exceedingly short period of the quantum mechanical oscillations, nor even on the $\sim 25ms$ period of the $\sim 40Hz$ scale of the classical oscillations, but on the scale of the *difference* of the periods of the two coupled modes, there will be, in this model, by virtue of the quantum mechanical effects associated with a rapid sequence of repeated probing actions, a strong tendency for the brain correlate of consciousness to follow the *expected* trajectory, in contrast to what would happen if only infrequent probing actions were made.

This analysis is based on a theory of the mind-brain connection that resolves in principle the basic interpretational problem of quantum theory, which is the problem of reconciling the classical character of our perceptions of the physical world with the very non-classical character of the state of the world generated by the combination of the Schroedinger equation and the uncertainty principle. The theory resolves also the central problem of the philosophy of mind, which is to reconcile the apparent causal power of our conscious efforts with the laws and principles of physics. The theory allows us to understand within the *dynamical framework* of orthodox (knowledge-associated-collapse) quantum physics the evident capacity of our conscious thoughts to influence our physical actions, and to become thereby integrated into the process of natural selection.

The discussion has focused so far on one very small region of the cortex, or rather on one pair of causally linked regions, with one member of the pair associated with one possible experience, and the other member of the pair associated with the rival possible experience. But each possible experience is presumably associated with an excitation in a large collection of such localized regions. Following the principles of quantum *field* theory the quantum state is represented by a *tensor product* of states associated with the individual tiny regions. Each of the two SHOs discussed above can be generalized to the *product of the set* of SHOs associated with the corresponding set of sub-regions, assumed non-overlapping. The SHO in each sub- region interacts with its own immediate environment, and, via neural connections, also to the SHOs in the other set. This latter coupling has been explicitly introduced into the Hamiltonian.

The mechanism under consideration here does not involve any effect that adds phases associated with different space-time locations. It does not involve any process that brings together phases associated with macroscopically separated sub-regions and reacts to the sum or difference of these phases. The quantum Zeno effect being examined here arises from the product—not the sum—of the effects associated with different sub-regions. Hence random phase factors attached—by virtue of interactions with differing individual local environments— to the above-described wave packets associated with different regions do not affect the quantum Zeno effect occurring in this model. Consequently, the usual arguments[4] to the effect that 'decoherence' effects will destroy *certain* quantum effects do not cover the quantum Zeno effect that occurs here, within the context of this von Neumann based simultaneous dynamical treatment of both the classical-quantum and the mind-brain connections.

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