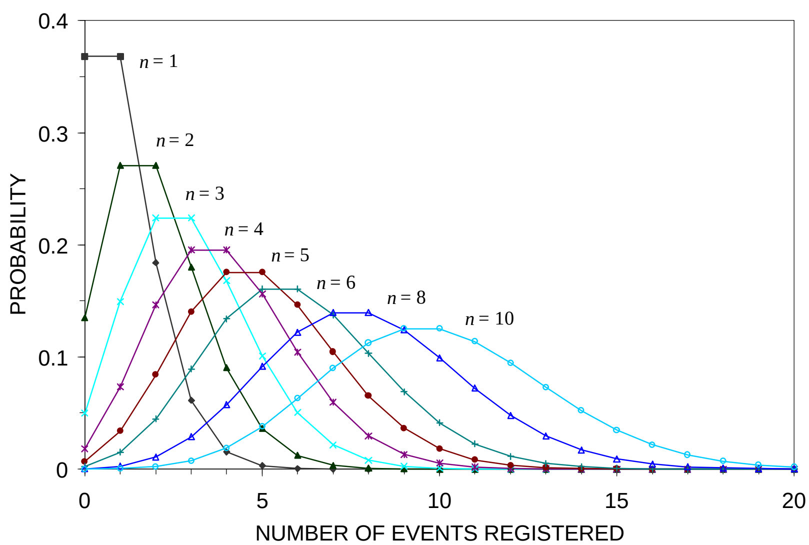
Physics 198, Spring Semester 1999 Helmuth Spieler
Introduction to Radiation Detectors and Electronics

Problem Set 4: Due on Tuesday, 23-Feb-99 at begin of lecture.
Discussion on Wednesday, 24-Feb-99 at 12 – 1 PM in 347 LeConte.
Office hours: Mondays, 3 – 4 PM in 420 LeConte

1. If within a given time interval the average number of events is $\overline{n}$, the probability of measuring *n* events is given by the Poisson distribution

$$P_n = \frac{(\overline{n})^n}{n!} e^{-\overline{n}}$$

Plot the probability distributions for $\overline{n}$ = 1, 2, 3, 5, 10. What is the probability of measuring zero events? Where does the distribution become approximately Gaussian?



Probability of measuring zero events:

n(average)	P(0)
1	0.37
2	0.14
3	0.05
5	0.007
10	4.5×10^{-5}

For deviations of about σ, $\overline{n}$ = 5 is sufficiently Gaussian. For larger deviations (> 2σ), average counts of 10 or more are necessary.

2. A silicon detector is fabricated using 300 μm thick *n*-type material with a donor concentration of 1.8×10^{12} cm^{-3}. The *p*-region is highly doped with a concentration of $>10^{16}$ cm^{-3}.

   a) What is the voltage required for full depletion?

   Since the carrier concentration in the *p*-region is much larger than in the *n*-region, the depletion region extends prediminantly into the bulk. The voltage required for full depletion

$$V_d = \frac{q_e N d^2}{2\varepsilon}$$

   For $N = 1.8\times10^{12}$ cm^{-3} and d= 300 μm, V_d= 123V.

   b) What are the collection times for electrons and holes when the detector is operated at 50, 100 and 200 V?

   At 50 and 100 V the detector is partially depleted, so the collection times are the same for both operating voltages. The dielectric relaxation time in *n*-type Si

$$\tau = 1.05\left[\frac{ns}{k\Omega\cdot cm}\right]\rho$$

   N= 1.8×10^{12} cm^{-3} corresponds to ρ = 2.31 kΩ·cm, so for 95% charge collection τ_n=3τ= 7.3 ns and τ_p=(1500/450)τ_n = 24.3 ns.

   At 200 V the detector is fully depleted, so the collection times for electrons and holes are

$$t_{cn} = \frac{W^2}{2\mu_n V_{di}}\ln\left(\frac{V_b + V_{di}}{V_b - V_{di}}\right)$$

   and

$$t_{cp} = \frac{W^2}{2\mu_p V_{di}}\ln\left(\frac{V_b + V_{di}}{V_b - V_{di}}\right)$$

   where V_{di} is the internal depletion voltage V_d+V_{bi}. For a depletion voltage of 123 V (and neglecting V_{bi}), t_{cn}= 3.5 ns and t_{cp}= 11.7 ns.

c) The detector is exposed to 20 keV x-rays. At 50 V the detector system yields a signal-to-noise ratio of 5. What is the noise level of the system? What is the signal-to-noise ratio at 100 and 200 V?

Since 3.6 eV are required to produce an electron-hole pair in Si, 20 keV produces a signal of 5555 electrons. Since the signal-to-noise ratio is 5, the noise level is 5560/5= 1111 el.

At 50 and 100 V the detector is partially depleted, so the ratio of detector capacitances

$$\frac{C_1}{C_2} = \sqrt{\frac{V_{b2}}{V_{b1}}}$$

Since the electronic noise is proportional to the capacitance, the noise level at 100 V will improve by

$$\frac{Q_n(100\text{V})}{Q_n(50\text{V})} = \sqrt{\frac{50}{100}} = \frac{1}{\sqrt{2}}$$

yielding 786 el.
At 200 V the detector is fully depleted, i.e. 300 μm thick. Since at 50 V the depletion depth is

$$d = \sqrt{\frac{2\varepsilon(V_d + V_{bi})}{q_e N}} = 191\,\mu\text{m}$$

so the capacitance at full depletion is 191/300= 0.64 smaller than at 50 V, so the noise reduces to 0.64 x 1111 el = 707 el.

d) Using the same electronics the detector is exposed to minimum ionizing particles, which have an energy loss dE/dx = 265 eV/μm. What is the signal at 50, 100 and 200 V? What is the signal-to-noise ratio?

The signal increases with linearly with depletion width, whereas the noise decreases linearly with depletion width. Thus, in partial depletion the signal-to-noise ratio increases with the square of depletion width. Since the depletion width increases with the square root of voltage, the signal-to-noise ratio will increase linearly with voltage until the detector is fully depleted. Then the signal and noise will remain constant.

V_b [V]	d [μm]	Signal [el]	Noise [el]	S/N
50	191	14060	1111	13
100	270	19875	786	25
200	300	22083	707	31

3. Consider an *RC* integrator driven by a voltage amplifier, i.e. the source impedance is low. The resistance R = 10 K and the capacitance C = 100 pF.

   a) Plot the magnitude and phase vs. frequency of the transmitted voltage. What is the signal bandwidth of this system? Calculate the noise bandwidth and compare it with the signal bandwidth.

   Output voltage

$$V_{out} = \frac{X_C}{R + X_C} V_{in} = \frac{-\frac{i}{\omega C}}{R - \frac{i}{\omega C}} V_{in}$$

$$V_{out} = \frac{1}{1 + i\omega RC} V_{in} = \frac{1 - i\omega RC}{1 + (\omega RC)^2} V_{in} \equiv \frac{1 - i\omega\tau}{1 + (\omega\tau)^2} V_{in}$$

Expressed as a magnitude and phase

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1 + (f / f_u)^2}} \quad \text{and} \quad \Theta = -\arctan\frac{f}{f_u}$$

where

$$f_u \equiv \frac{1}{2\pi RC}$$

The input voltage drives the current flow through the capacitor. Since the voltage at the output is the integral of the current, its phase lags relative to the input.

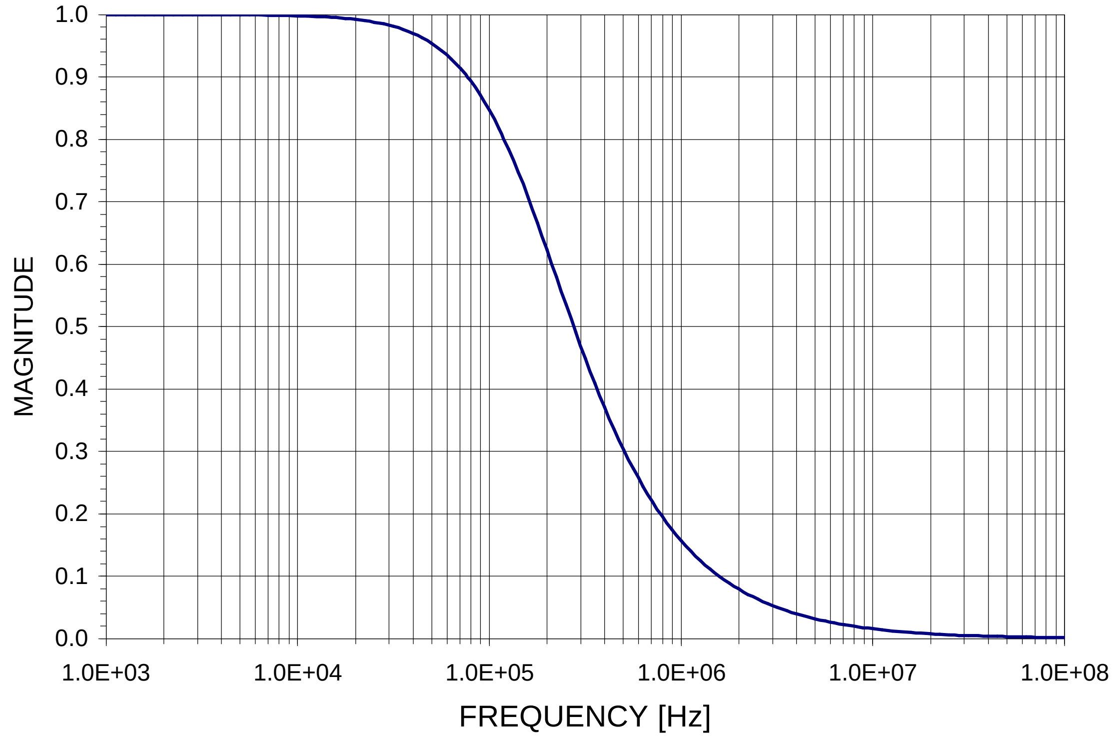
At low frequencies $\omega << 1/\tau$ $(f << f_u)$: $V_{out} \approx V_{in}$

At high frequencies, $\omega >> 1/\tau$ $(f >> f_u)$: $V_{out} \approx -i\frac{1}{\omega\tau} \cdot V_{in}$

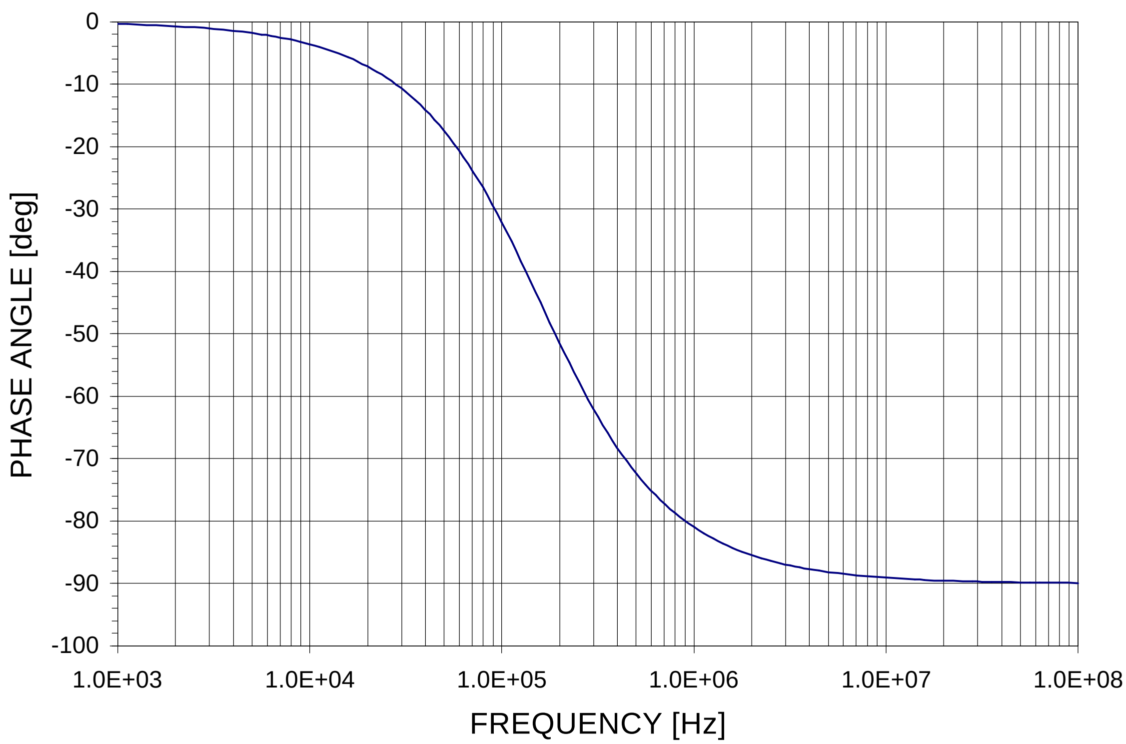
⇒ the *RC* integrator is a “low-pass” filter, i.e. it transmits frequencies below the cutoff frequency $f_u = 1/2\pi RC$.

Noise Bandwidth: $$\Delta f_n = \int_0^\infty \left|\frac{V_{out}}{V_{in}}\right|^2 df = \int_0^\infty \frac{1}{1 + (f / f_u)^2} df = \frac{1}{4RC}$$

Signal Bandwidth $$f_u = \frac{1}{2\pi RC} = \frac{2}{\pi}\Delta f_n \quad \Rightarrow \quad \Delta f_n = \frac{\pi}{2} f_u$$



MAGNITUDE
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
1.0E+03
1.0E+04
1.0E+05
1.0E+06
1.0E+07
1.0E+08
FREQUENCY [Hz]



PHASE ANGLE [deg]
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
1.0E+03
1.0E+04
1.0E+05
1.0E+06
1.0E+07
1.0E+08
FREQUENCY [Hz]

b) Derive the pulse response to a step input with negligible rise time. What is the time constant of the output pulse? What is the relationship between the time constant and the rise time measured as a difference between the times when the output attains 10% and 90% of the peak signal?

Pulse response:

input voltage = voltage across C + voltage across R

$$\frac{dV_{in}(t)}{dt} = \frac{1}{C}\frac{dQ}{dt} + \frac{dV_R(t)}{dt}$$

$$\frac{dV_{in}(t)}{dt} = \frac{1}{C}i(t) + R\frac{di(t)}{dt}$$

For a step input $V_{in} = 0$ for $t < 0$
$V_{in} = V_i$ for $t \geq 0$

At t ≥ 0, $dV_{in}(t)/dt = 0$, so

$$\frac{1}{C}i(t) + R\frac{di(t)}{dt} = 0 \qquad \Rightarrow \qquad -\frac{1}{RC}dt = \frac{1}{i(t)}dt$$

Setting $\tau = RC$ and integrating yields

$$-\frac{t}{\tau} = \ln i(t) + \ln K \qquad \Rightarrow \qquad i(t) = \frac{1}{K}e^{-t/\tau}$$

The output voltage

$$V_o(t) = \frac{1}{C}\int i(t)dt = \frac{1}{KC}\int_0^t e^{-t/\tau}dt = -\frac{\tau}{KC}\cdot e^{-t/\tau}\Big|_0^t$$

$$V_o(t) = \frac{RC}{KC}\left(1 - e^{-t/\tau}\right)$$

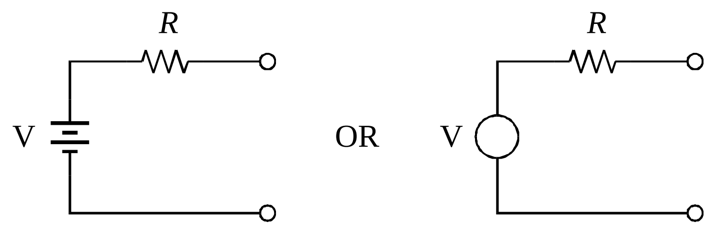
Since at $t = \infty$, $V_o = V_i$: $K = R/V_i$

$$V_o(t) = V_i\left(1 - e^{-t/\tau}\right)$$

The time constant $\tau = RC = 10^4 \text{ x } 10^{-10} = 10^{-6} = 1$ µs.

The output signal attains 10% of its final value at 0.1τ and 90% at 2.3τ, so the 10-90% rise time is $2.2\tau = 2.2$ µs.

4. Although an ideal voltage source provides an output voltage that is independent of the current drawn, real voltage sources exhibit a finite source resistance. Over a restricted range of load currents voltage sources can be represented by the equivalent circuit,

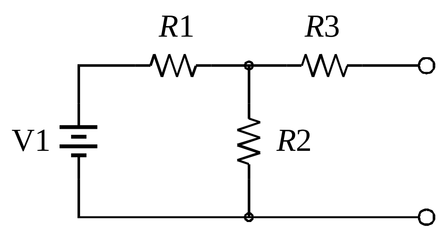


where the battery is an ideal voltage source (i.e. with zero source resistance) providing the voltage V (Thévenin equivalent circuit).

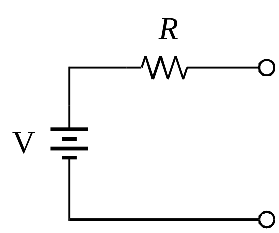
a) What is the dependence of output voltage on load current?

$$V_{out} = V - I_{load} R$$

b) Draw the equivalent circuit of the following network showing the component values.



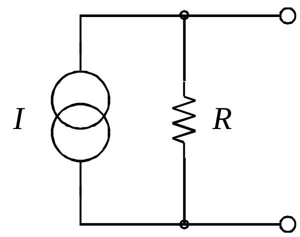
Equivalent Circuit:



$$V = V1 \frac{R2}{R1 + R2}$$

$$R = \frac{R1R2}{R1 + R2} + R3$$

c) The equivalent circuit of a voltage source can also be constructed with a current source (Norton equivalent circuit). This is useful in nodal analyses where currents must be summed. An ideal current source forces a constant current into the load, independent of the load resistance (i.e. independent of the voltage across the load).



What is the equivalent circuit of the network shown in b) when using a current source?

For an external load drawing the current I_L, the output voltage of the Norton equivalent circuit is

$$V_{out} = (I - I_L)R = IR - I_L R$$

The voltage source in problem b) has the *I*-*V* characteristic

$$V_{out} = V1\frac{R2}{R1+R2} - I_L\left(\frac{R1R2}{R1+R2} + R3\right)$$

Comparison yields

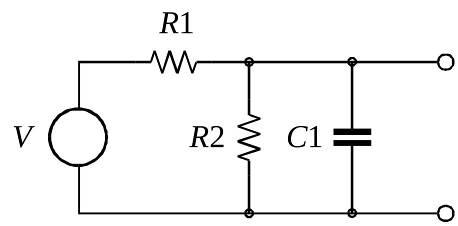
$$R = \frac{R1R2}{R1+R2} + R3$$

and

$$I \cdot \left(\frac{R1R2}{R1+R2} + R3\right) = V1\frac{R2}{R1+R2}$$

$$I = V1\frac{R2}{R1R2 + R3(R1+R2)}$$

5. In the circuit below the voltage generator provides either an alternating voltage (sine wave) in problem a) or a step impulse in problem b).



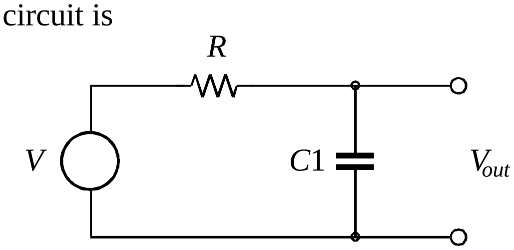
a) Plot the magnitude and phase of the output voltage as the frequency of the voltage generator is varied.

b) What is the output waveform if the source generates a voltage step?

Hint: First derive the equivalent circuit for the network V, $R1$, $R2$ and then consider the effect of $C1$..

Solutions:

a) $R1$ and $R2$ can be combined into a single series resistor $R = \frac{R1R2}{R1+R2}$, so the equivalent circuit is



This is a conventional RC integrator, as analyzed in Problem 3. The upper cutoff frequency is

$$f_u = \frac{1}{2\pi RC1}$$

and the magnitude and phase vs. frequency are as given above,

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1+(f/f_u)^2}} \quad \text{and} \quad \Theta = -\arctan\frac{f}{f_u}$$

b) What is the output waveform if the source generates a voltage step?

$$V_{out} = V\left(1-e^{-t/\tau}\right)$$

where $\tau = RC1$.