Noise Analysis in the Time Domain

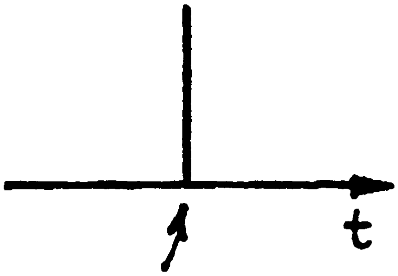
What pulse shapes have a frequency spectrum corresponding to typical noise sources?

1. voltage noise

The frequency spectrum at the input of the detector system is “white”, i.e.

$$\frac{dA}{df} = \text{const.}$$

This is the spectrum of a δ impulse:



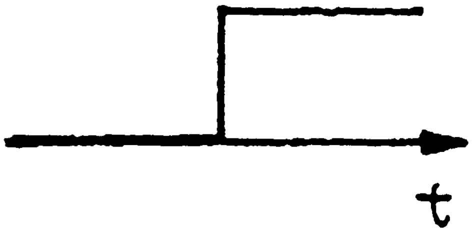
inifinitesimally narrow,
but area= 1.

2. current noise

The spectral density is inversely proportional to frequency, i.e.

$$\frac{dA}{df} \propto \frac{1}{f}$$

This is the spectrum of a step impulse:



- Input noise can be considered as a sequence of δ and step pulses whose rate determines the noise level.

- The shape of the primary noise pulses is modified by the pulse shaper:

  δ pulses become longer,

  step pulses are shortened.

- The noise level at a given measurement time T_m is determined by the cumulative effect (superposition) of all noise pulses occurring prior to T_m .

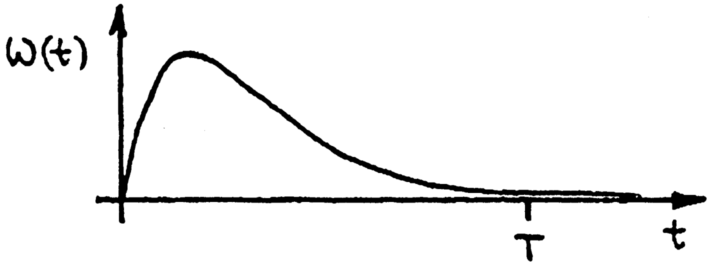
- Their individual contributions at $t = T_m$ are described by the shaper's "weighting function" $W(t)$.

References:

V. Radeka, Nucl. Instr. and Meth. **99** (1972) 525
V. Radeka, IEEE Trans. Nucl. Sci. **NS-21** (1974) 51
F.S. Goulding, Nucl. Instr. and Meth. **100** (1972) 493
F.S. Goulding, IEEE Trans. Nucl. Sci. **NS-29** (1982) 1125

Consider a single noise pulse occurring in a short time interval dt at a time T prior to the measurement. The amplitude at $t= T$ is

$$a_n = W(T)$$



If, on the average, $n_n\, dt$ noise pulses occur within dt, the fluctuation of their cumulative signal level at $t= T$ is proportional to

$$\sqrt{n_n dt}$$

The magnitude of the baseline fluctuation is

$$\sigma_n^2(T) \propto n_n \left[W(t)\right]^2 dt$$

For all noise pulses occurring prior to the measurement

$$\sigma_n^2 \propto n_n \int_0^\infty \left[W(t)\right]^2 dt$$

where

n_n	determines the magnitude of the noise

and

$\int_0^\infty \left[W(t)\right]^2 dt$	describes the noise characteristics of the shaper – the “noise index”

The Weighting Function

a) current noise

$W_i(t)$ is the shaper response to a step pulse, i.e. the “normal” output waveform.

b) voltage noise

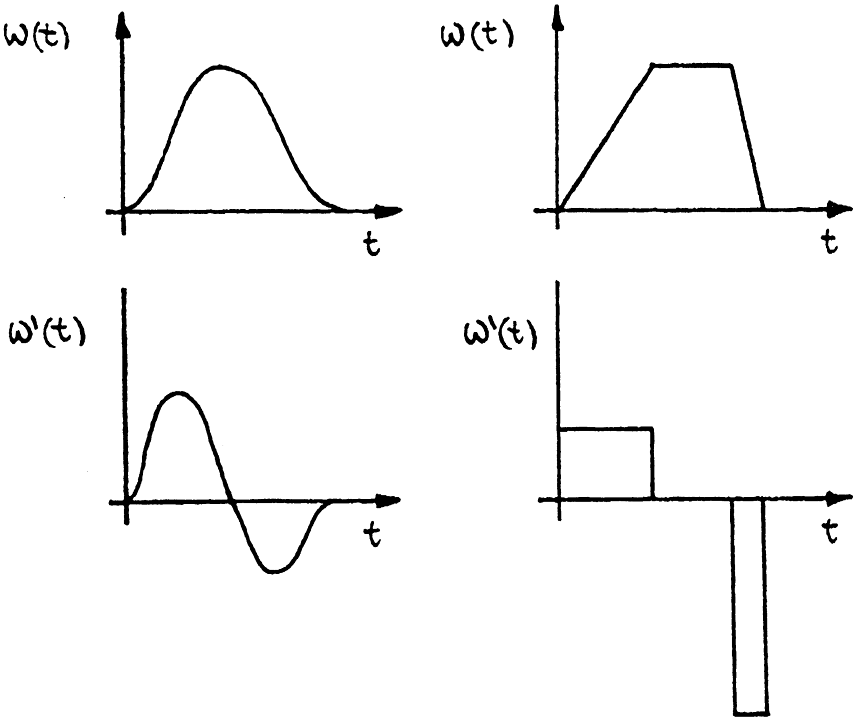
$$W_v(t) = \frac{d}{dt} W_i(t) \equiv W'(t)$$

(Consider a δ pulse as the superposition of two step pulses of opposite polarity and spaced inifinitesimally in time)

Examples: 1. Gaussian 2. Trapezoid

current (“step”) noise

voltage (“delta”) noise



Goal: Minimize overall area to reduce current noise contribution
Minimize derivatives to reduce voltage noise contribution

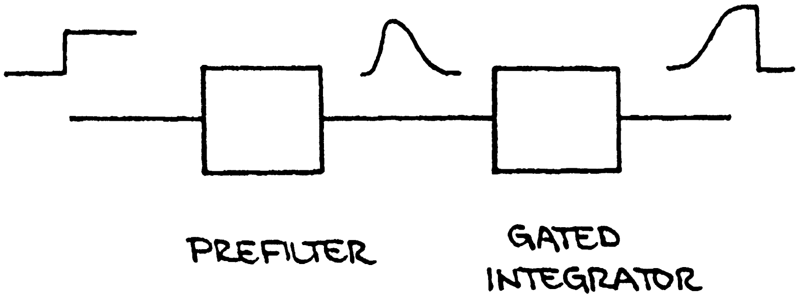
⇒ For a given pulse duration a symmetrical pulse provides the best noise performance.
Linear transitions minimize voltage noise contributions.

Time-Variant Shapers

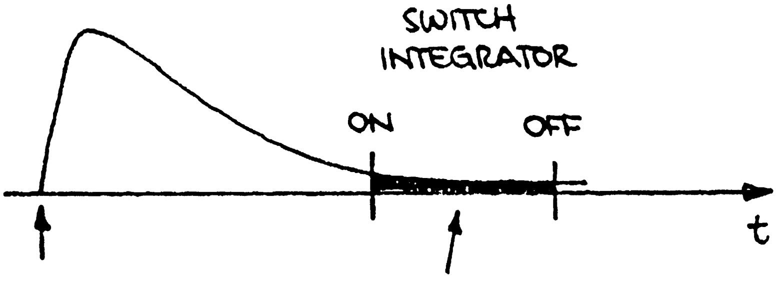
Example: gated integrator with prefilter

The gated integrator integrates the input signal during a selectable time interval (the “gate”).

In this example, the integrator is switched on prior to the signal pulse and switched off after a fixed time interval, selected to allow the output signal to reach its maximum.



Consider a noise pulse occurring prior to the “on time” of the integrator.



occurrence of the noise pulse

contribution of noise pulse to integrator output

For W_1 = weighting function of the <u>time-invariant</u> prefilter

W_2 = weighting function of the <u>time-variant</u> stage

the overall weighting function is obtained by convolution

$$W(t) = \int_{-\infty}^{\infty} W_2(t') \cdot W_1(t-t')\, dt'$$

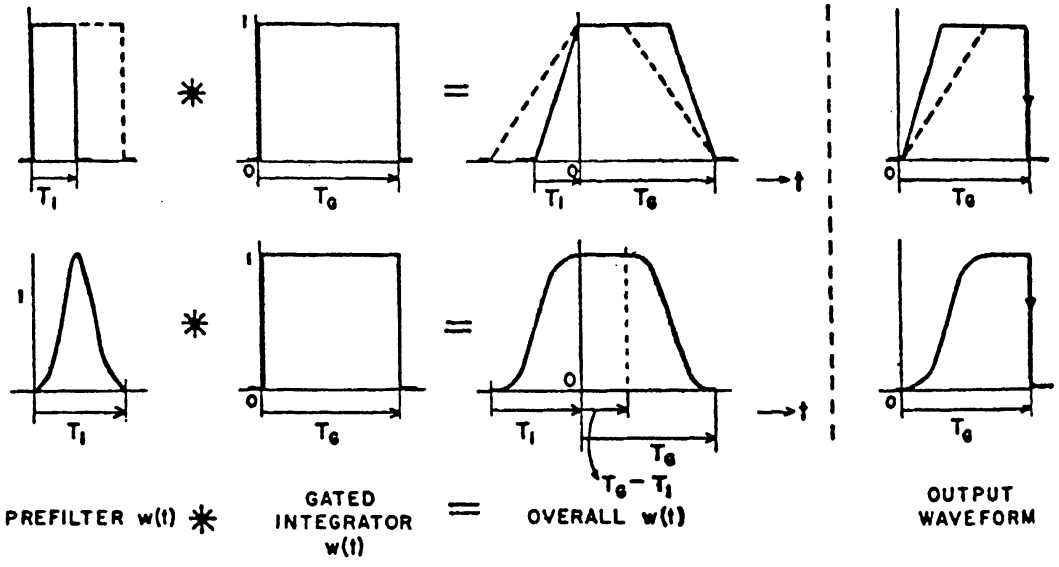
Weighting function for current ("step") noise: $W(t)$

Weighting function for voltage ("delta") noise: $W'(t)$

Example

Time-invariant prefilter feeding a gated integrator

(from Radeka, IEEE Trans. Nucl. Sci. **NS-19** (1972) 412)



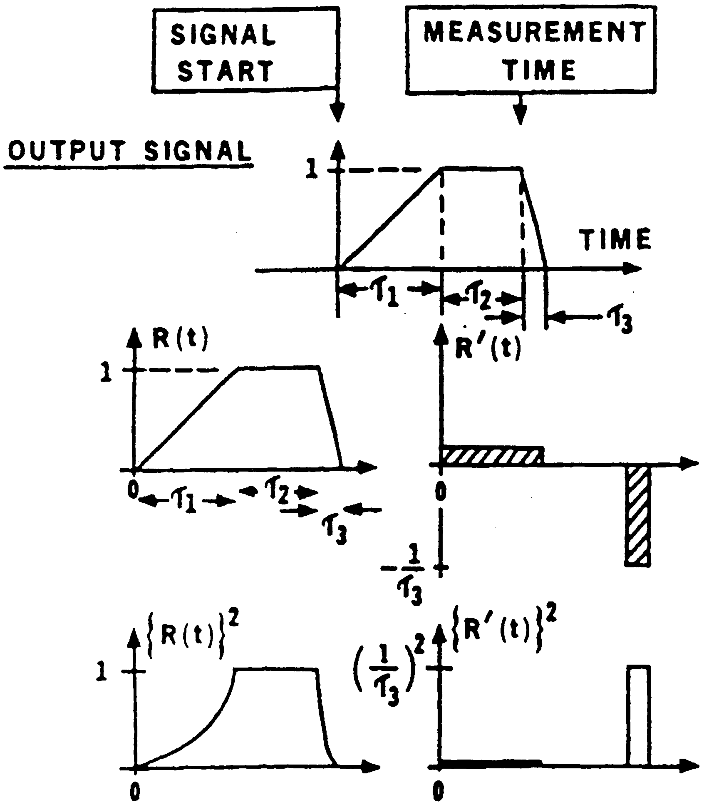
Comparison between a time-invariant and time-variant shaper

(from Goulding, NIM **100** (1972) 397)

Example: trapezoidal shaper

Duration= 2 µs
Flat top= 0.2 µs

1. Time-Invariant Trapezoid



Current noise

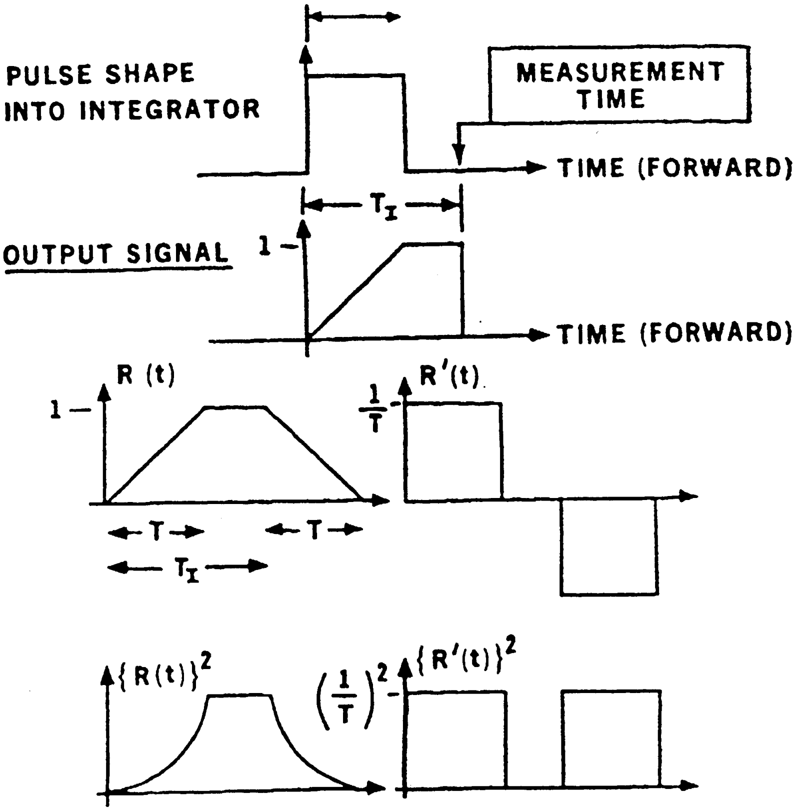
$$N_i^2 = \int_0^\infty [W(t)]^2 dt = \int_0^{\tau_1} \left(\frac{t}{\tau_1}\right)^2 dt + \int_{\tau_1}^{\tau_2} (1)^2 dt + \int_{\tau_2}^{\tau_3} \left(\frac{t}{\tau_3}\right)^2 dt = \tau_2 + \frac{\tau_1 + \tau_3}{3}$$

Voltage noise

$$N_v^2 = \int_0^\infty [W'(t)]^2 dt = \int_0^{\tau_1} \left(\frac{1}{\tau_1}\right)^2 dt + \int_{\tau_2}^{\tau_3} \left(\frac{1}{\tau_3}\right)^2 dt + = \frac{1}{\tau_1} + \frac{1}{\tau_3}$$

Minimum for $\tau_1 = \tau_3$ (symmetry!) ⇒ $N_i^2 = 0.8$, $N_v^2 = 2.2$

Gated Integrator Trapezoidal Shaper



Current Noise

$$N_i^2 = 2\int_0^T \left(\frac{t}{T}\right)^2 dt + \int_T^{T_I - T} (1)^2 dt = T_I - \frac{T}{3}$$

Voltage Noise

$$N_v^2 = 2\int_0^T \left(\frac{1}{T}\right)^2 dt = \frac{2}{T}$$

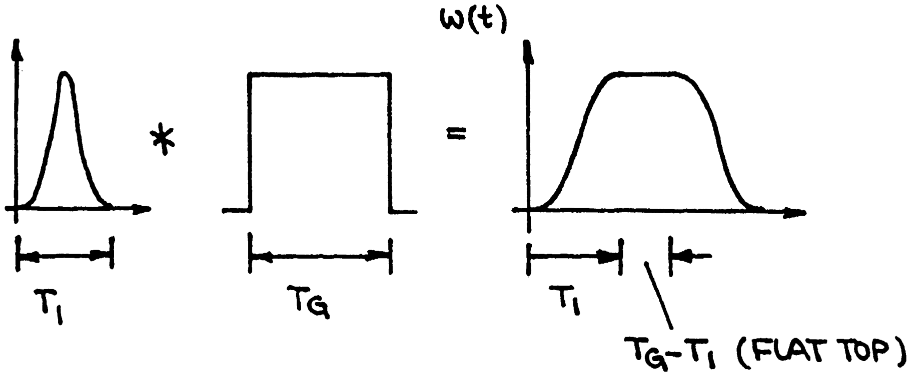
⇒ time-variant shaper N_i^2= 1.4, N_v^2= 1.1

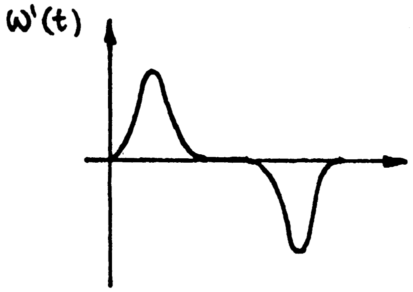
time-invariant shaper N_i^2= 0.8, N_v^2= 2.2

time-variant trapezoid has more current noise, less voltage noise

Interpretation of Results

Example: gated integrator





Current Noise

$$Q_{ni}^2 \propto \int [W(t)]^2 \, dt$$

Increases with T_1 and T_G (i.e. width of $W(t)$)

(more noise pulses accumulate within width of $W(t)$)

Voltage Noise

$$Q_{nv}^2 \propto \int [W'(t)]^2 \, dt$$

Increases with the magnitude of the derivative of $W(t)$

(steep slopes → large bandwidth — *determined by prefilter*)

Width of flat top irrelevant

(δ response of prefilter is bipolar: net= 0)

Quantitative Assessment of Noise in the Time Domain

(see Radeka, IEEE Trans. Nucl. Sci. **NS-21** (1974) 51)

$$Q_n^2 = \frac{1}{2}\, i_n^2 \int\limits_{-\infty}^{\infty} [W(t)]^2\, dt \;+\; \frac{1}{2}\, C_i^2\, v_n^2 \int\limits_{-\infty}^{\infty} [W'(t)]^2\, dt$$

↑ current noise ↑ voltage noise

Q_n= equivalent noise charge [C]

i_n= input current noise spectral density [A/√Hz]

v_n= input voltage noise spectral density [V/√Hz]

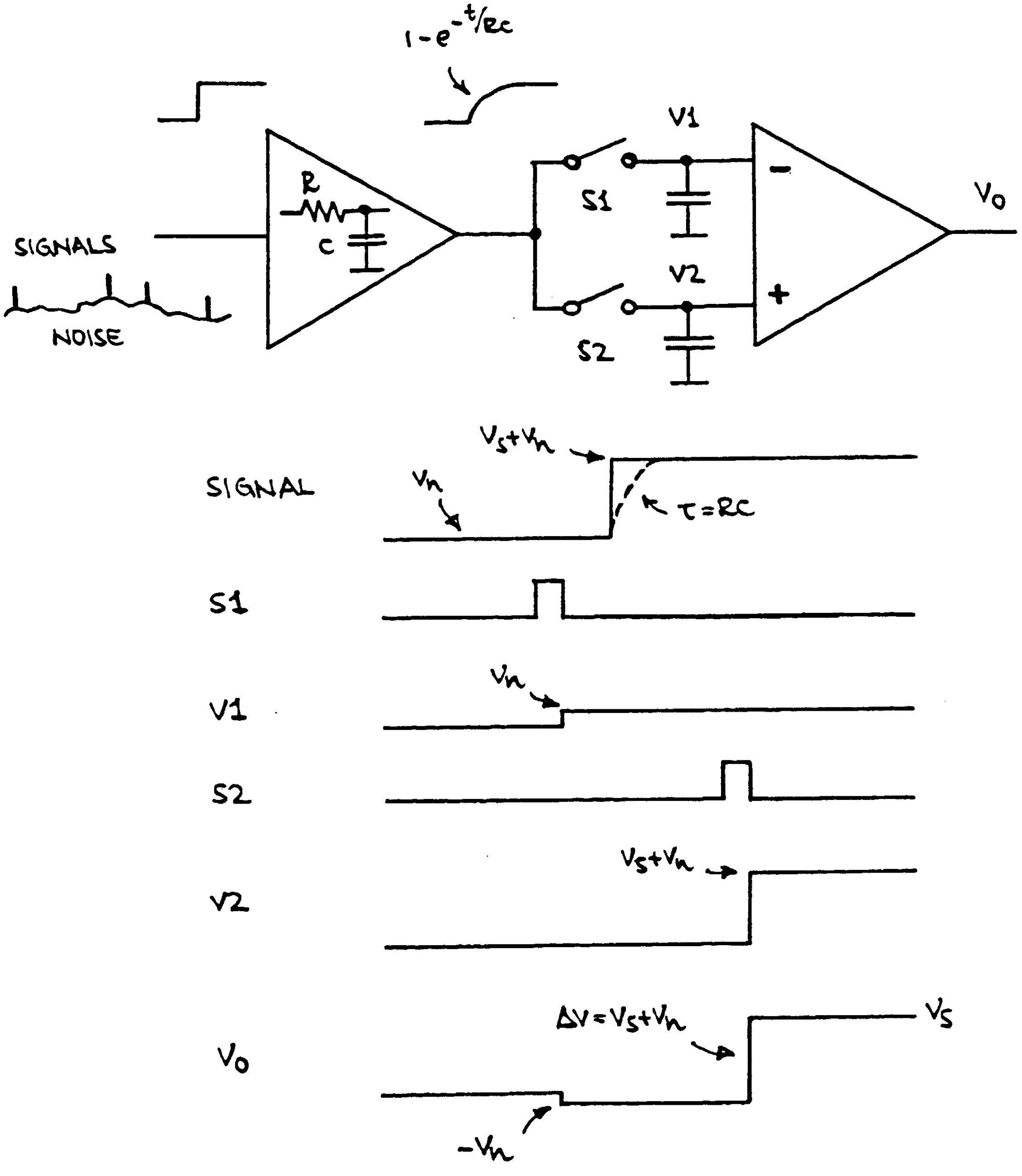
C_i= total capacitance at input

$W(t)$ normalized to unit input step response

or rewritten in terms of a characteristic time $t \rightarrow T/\,t$

$$Q_n^2 = \frac{1}{2}\, i_n^2\, T \int\limits_{-\infty}^{\infty} [W(t)]^2\, dt \;+\; \frac{1}{2}\, C_i^2\, v_n^2\, \frac{1}{T} \int\limits_{-\infty}^{\infty} [W'(t)]^2\, dt$$

Correlated Double Sampling



1. Signals are superimposed on a (slowly) fluctuating baseline
2. To remove baseline fluctuations the baseline is sampled prior to the arrival of a signal.
3. Next, the signal + baseline is sampled and the previous baseline sample subtracted to obtain the signal

1. Current Noise

Current (shot) noise contribution:

$$Q_{ni}^2 = \frac{1}{2} i_n^2 \int_{-\infty}^{\infty} [W(t)]^2 \, dt$$

Weighting function (T= time between samples):

$$\begin{aligned} t < 0: &\quad W(t) = 0 \\ 0 \le t \le T: &\quad W(t) = 1 - e^{-t/\tau} \\ t > T: &\quad W(t) = e^{-(t-T)/\tau} \end{aligned}$$

Current noise coefficient

$$F_i = \int_{-\infty}^{\infty} [W(t)]^2 \, dt$$

$$F_i = \int_0^T \left(1 - e^{-t/\tau}\right)^2 dt + \int_T^{\infty} e^{-2(t-T)/\tau} \, dt$$

$$F_i = \left(T + \frac{\tau}{2} e^{-T/\tau} - \frac{\tau}{2} e^{-2T/\tau} \right) + \frac{\tau}{2}$$

so that the equivalent noise charge

$$Q_{ni}^2 = \frac{1}{2} i_n^2 \left[T + \frac{\tau}{2} \left(e^{-T/\tau} - e^{-2T/\tau} + 1 \right) \right]$$

$$Q_{ni}^2 = i_n^2 \; \tau \quad \frac{1}{4} \left(\frac{2T}{\tau} + e^{-T/\tau} - e^{-2T/\tau} + 1 \right)$$

Reality Check 1:

Assume that the current noise is pure shot noise

$$i_n^2 = 2q_e I$$

so that

$$Q_{ni}^2 = q_e I \tau \, \frac{1}{2}\left(\frac{2T}{\tau} + e^{-T/\tau} - e^{-2T/\tau} + 1 \right)$$

Consider the limit Sampling Interval >> Rise Time, $T >> \tau$:

$$Q_{ni}^2 \approx q_e I \cdot T$$

or expressed in electrons

$$Q_{ni}^2 \approx \frac{q_e I \cdot T}{q_e^2} = \frac{I \cdot T}{q_e}$$

$$Q_{ni} \approx \sqrt{N_i}$$

where N_i is the number of electrons “counted” during the sampling interval T.

2. Voltage Noise

Voltage Noise Contribution

$$Q_{nv}^2 = \frac{1}{2} C_i^2 v_n^2 \int_{-\infty}^{\infty} [W'(t)]^2 \, dt$$

Voltage Noise Coefficient

$$F_v = \int_{-\infty}^{\infty} [W'(t)]^2 \, dt$$

$$F_v = \int_0^T \left(\frac{1}{\tau} e^{-t/\tau} \right)^2 dt + \int_T^{\infty} \left(\frac{1}{\tau} e^{-2(t-T)/\tau} \right)^2 dt$$

$$F_v = \frac{1}{2\tau} \left(1 - e^{-2T/\tau} \right) + \frac{1}{2\tau}$$

$$F_v = \frac{1}{2\tau} \left(2 - e^{-2T/\tau} \right)$$

so that the equivalent noise charge

$$Q_{nv}^2 = C_i^2 \, v_n^2 \, \frac{1}{\tau} \, \frac{1}{4} \left(2 - e^{-2T/\tau} \right)$$

Reality Check 2:

In the limit $T >> \tau$:

$$Q_{nv}^2 = C_i^2 \cdot v_n^2 \cdot \frac{1}{2\tau}$$

Compare this with the noise on an RC low-pass filter alone (i.e. the voltage noise at the output of the pre-filter):

$$Q_n^2(RC) = C_i^2 \cdot v_n^2 \cdot \frac{1}{4\tau}$$

(see the discussion on noise bandwidth)

so that

$$\frac{Q_n(\text{double sample})}{Q_n(RC)} = \sqrt{2}$$

If the sample time is sufficiently large, the noise samples taken at the two sample times are uncorrelated, so the two samples simply add in quadrature.

3. Signal Response

The preceding calculations are only valid for a signal response of unity, which is valid at $T >> \tau$.

For sampling times T of order τ or smaller one must correct for the reduction in signal amplitude at the output of the prefilter

$$V_s / V_i = 1 - e^{-T/\tau}$$

so that the equivalent noise charge due to the current noise becomes

$$Q_{ni}^2 = i_n^2 \; \tau \quad \frac{\dfrac{2T}{\tau} + e^{-T/\tau} - e^{-2T/\tau} + 1}{4\left(1 - e^{-T/\tau}\right)^2}$$

The voltage noise contribution is

$$Q_{nv}^2 = C_i^2 \; v_n^2 \; \frac{1}{\tau} \; \frac{2 - e^{-2T/\tau}}{4\,(1 - e^{-T/\tau})^2}$$

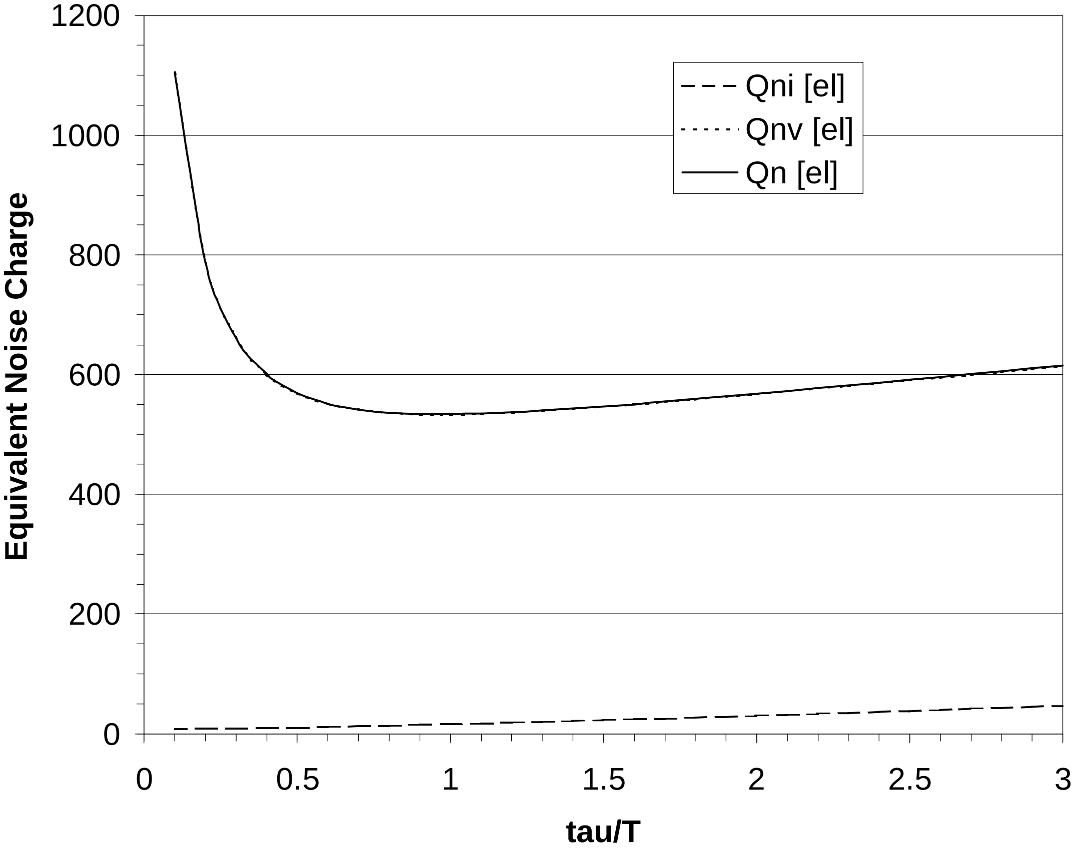
and the total equivalent noise charge

$$Q_n = \sqrt{Q_{ni}^2 + Q_{nv}^2}$$

Optimization

1. Noise current negligible

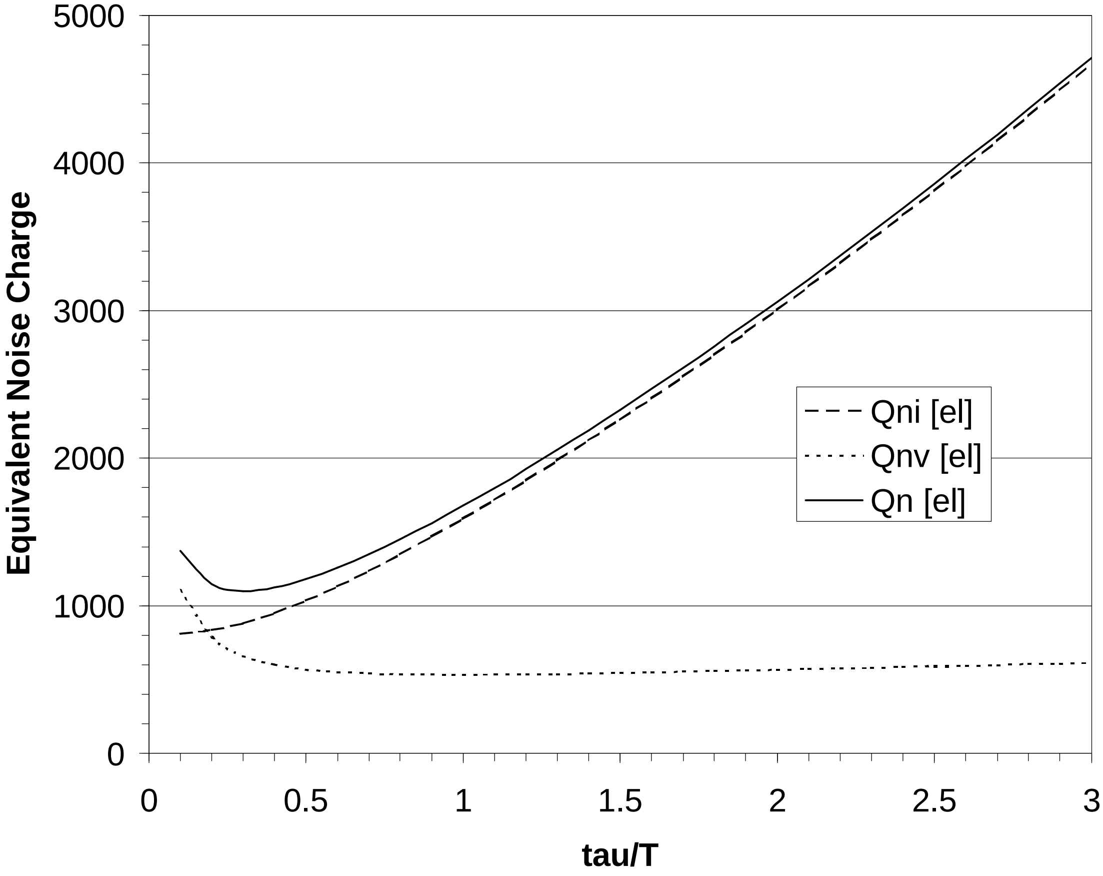
Parameters: T= 100 ns
C_d= 10 pF
v_n= 2.5 nV/√Hz
→ i_n= 6 fA/√Hz (I_b= 0.1 nA)



Noise attains shallow minimum for $\tau = T$.

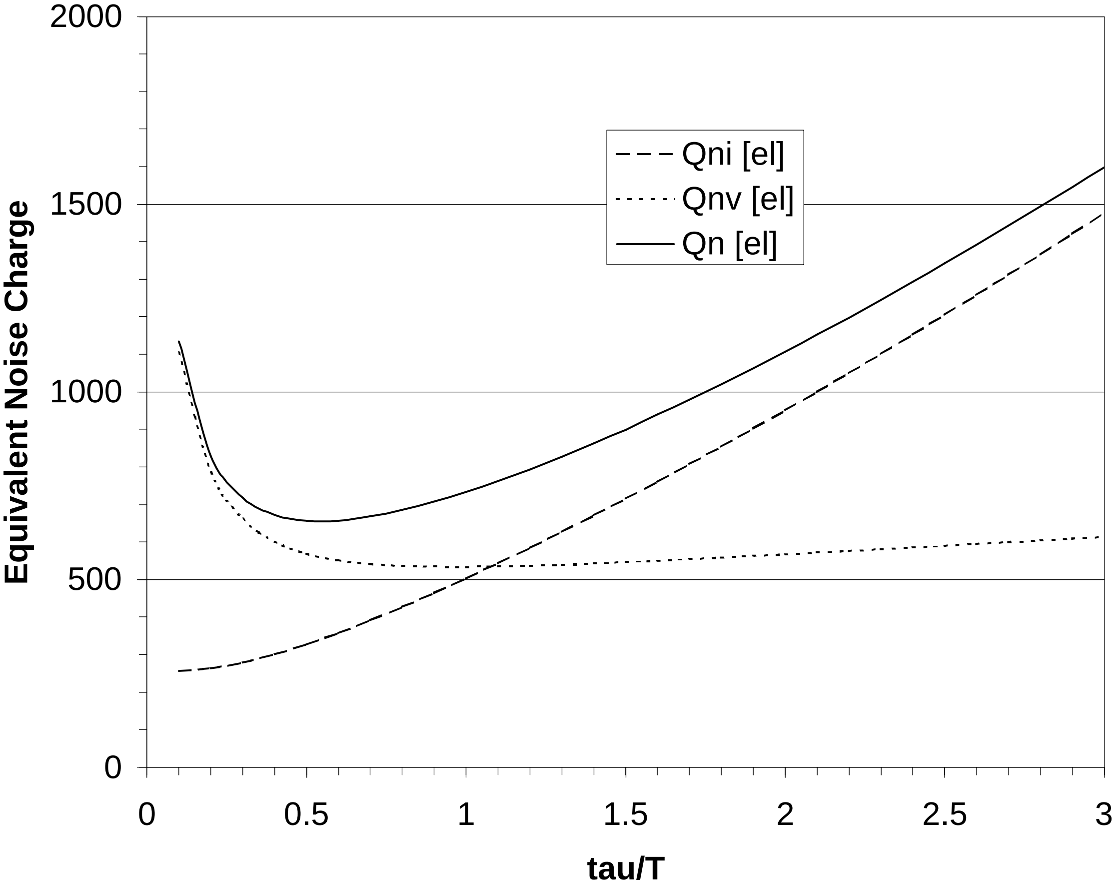
2. Significant current noise contribution

Parameters: T= 100 ns
C_d= 10 pF
v_n= 2.5 nV/√Hz
→ i_n= 0.6 pA/√Hz (I_b= 1 μA)



Noise attains minimum for $\tau = 0.3\ T$.

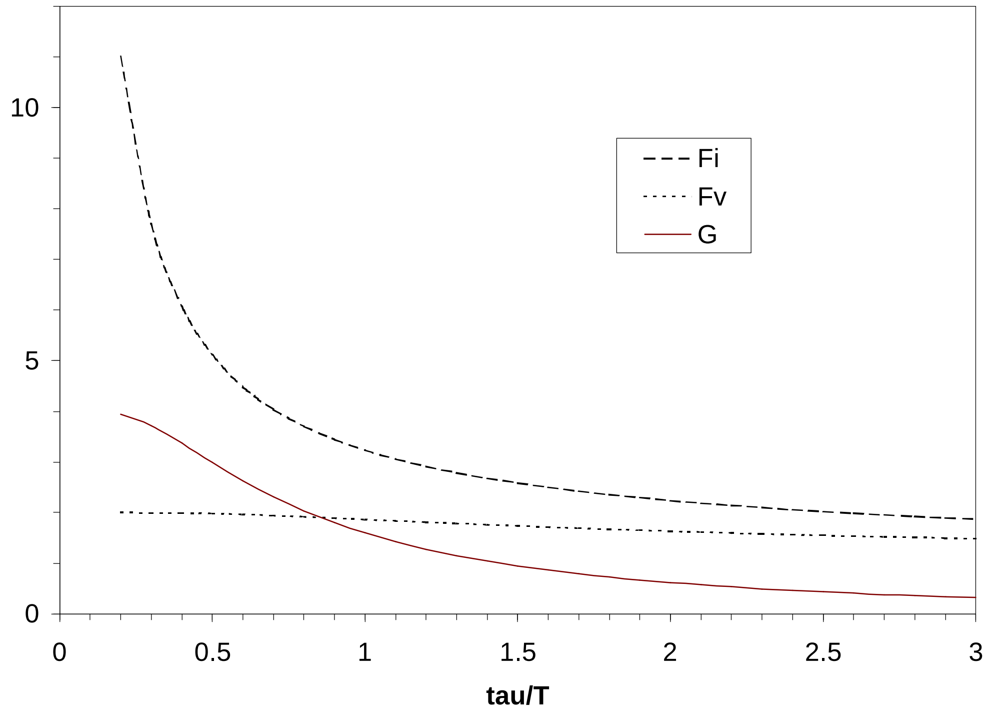
Parameters: T= 100 ns
C_d= 10 pF
v_n= 2.5 nV/√Hz
→ i_n= 0.2 pA/√Hz (I_b= 100 nA)



Noise attains minimum for $\tau = 0.5\ T$.

3. Form Factors F_i, F_v and Signal Gain G vs. τ / T

Note: In this plot the form factors F_i, F_v are not yet corrected by the gain G.



The voltage noise coefficient is practically independent of τ / T .

Voltage contribution to noise charge dominated by C_i / τ .

The current noise coefficient increases rapidly at small τ / T .

At small τ / T (large T) the contribution to the noise charge increases because the integration time is larger.

The gain dependence increases the equivalent noise charge with increasing τ / T (as the gain is in the denominator).

Summary

Two basic noise mechanisms: input noise current i_n
input noise voltage v_n

Equivalent Noise Charge:

$$Q_n^2 = \underset{\substack{\uparrow \\ \text{front} \\ \text{end}}}{i_n^2} \; \underset{\substack{\uparrow \\ \text{shaper}}}{T_s F_i} + \underset{\substack{\uparrow \\ \text{front} \\ \text{end}}}{C_i^2 v_n^2} \; \underset{\substack{\uparrow \\ \text{shaper}}}{\frac{F_v}{T_s}}$$

where	T_s	Characteristic shaping time (*e.g.* peaking time)
	F_i, F_v	"Form Factors" that are determined by the shape of the pulse. They can be calculated in the frequency or time domain.
	C_i	Total capacitance at the input node (detector capacitance + input capacitance of preamplifier + stray capacitance + …)

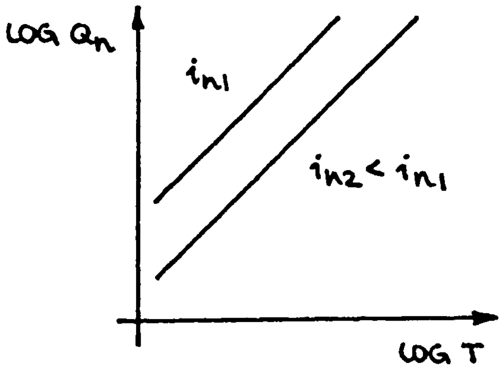
- Current noise contribution increases with T
- Voltage noise contribution decreases with increasing T

  Only for “white” voltage noise sources + capacitive load

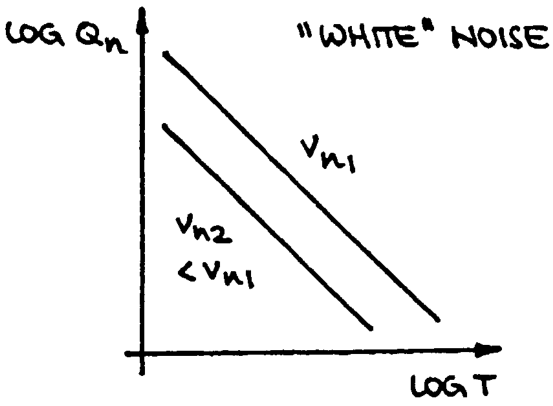
  “$1/f$” voltage noise contribution constant in T

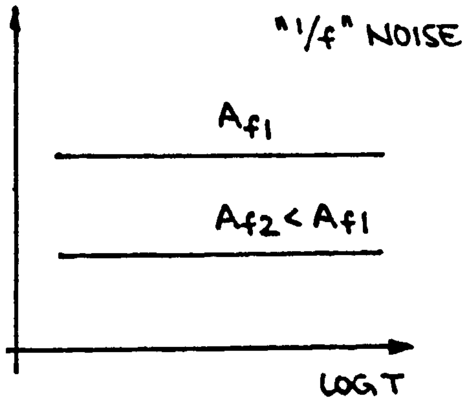
1. Equivalent Noise Charge vs. Pulse Width

Current Noise vs. T

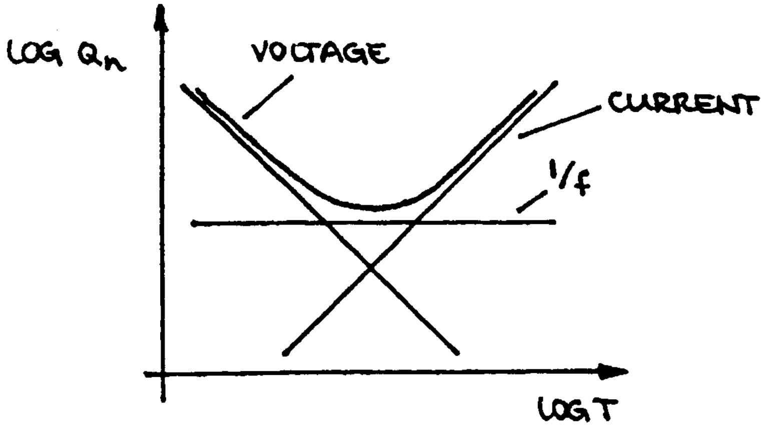


Voltage Noise vs. T





Total Equivalent Noise Charge



2. Equivalent Noise Charge vs. Detector Capacitance ($C_i = C_d + C_a$)

$$Q_n = \sqrt{i_n^2 F_i T + (C_d + C_a)^2 v_n^2 F_v \frac{1}{T}}$$

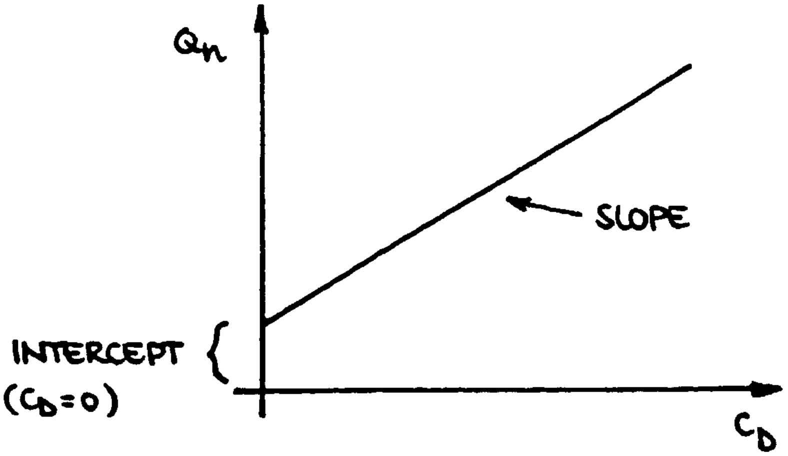
$$\frac{dQ_n}{dC_d} = \frac{2C_d v_n^2 F_v \frac{1}{T}}{\sqrt{i_n^2 F_i T + (C_d + C_a)^2 v_n^2 F_v \frac{1}{T}}}$$

If current noise $i_n^2 F_i T$ is negligible

$$\frac{dQ_n}{dC_d} \approx \underset{\substack{\uparrow \\ \text{input} \\ \text{stage}}}{2v_n} \cdot \underset{\substack{\uparrow \\ \text{shaper}}}{\sqrt{\frac{F_v}{T}}}$$

Zero intercept

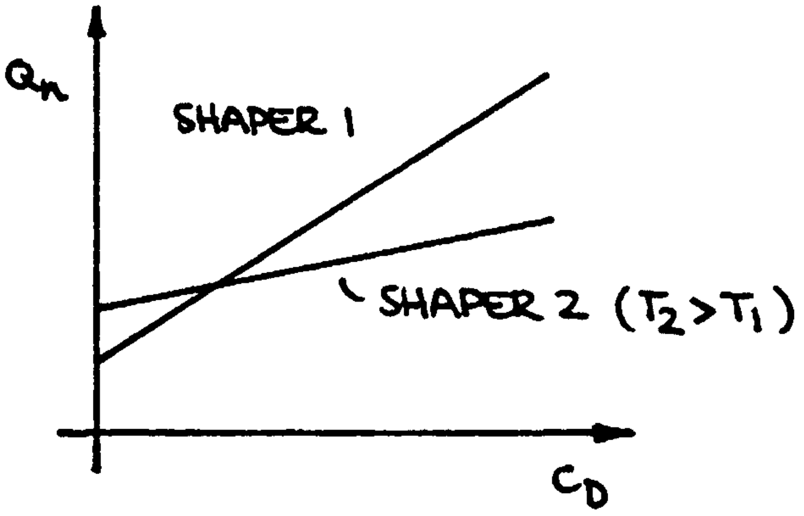
$$Q_n\big|_{C_d=0} = C_a v_n \sqrt{F_v / T}$$



Noise slope is a convenient measure to compare preamplifiers and predict noise over a range of capacitance.

Caution: both noise slope and zero intercept depend on *both the preamplifier and the shaper*

Same preamplifier, but different shapers:



Caution: Current noise contribution may be negligible at high detector capacitance, but not for $C_d=0$.

$$Q_n\Big|_{C_d=0} = \sqrt{i_n^2 F_i T + C_a^2 v_n^2 F_v / T}$$