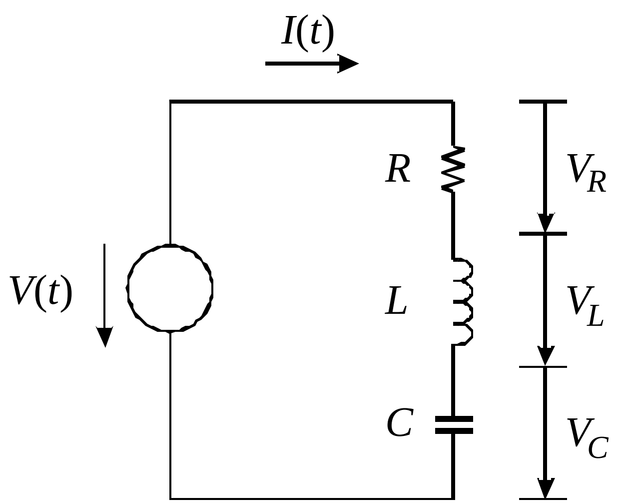
Appendices

Appendix 1

Phasors and Complex Algebra in Electrical Circuits

Consider the *RLC* circuit



$$V = V_R + V_L + V_C$$

$$V = IR + L\frac{dI}{dt} + \frac{Q}{C}$$

$$\frac{dV}{dt} = \frac{dI}{dt}R + L\frac{d^2 I}{dt^2} + \frac{I}{C}$$

Assume that $\quad V(t) = V_0 e^{\mathbf{i}\omega t} \quad$ and $\quad I(t) = I_0 e^{\mathbf{i}(\omega t + \varphi)}$

$$\mathbf{i}\omega V_0 e^{\mathbf{i}\omega t} = \mathbf{i}\omega R I_0 e^{\mathbf{i}(\omega t - \varphi)} - \omega^2 L I_0 e^{\mathbf{i}(\omega t - \varphi)} + \frac{1}{C} I_0 e^{\mathbf{i}(\omega t - \varphi)}$$

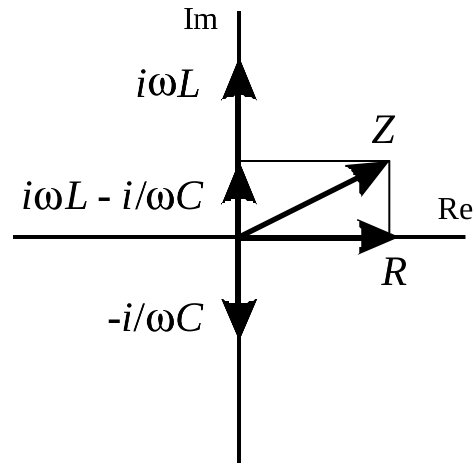
$$\frac{V_0}{I_0} e^{\mathbf{i}\varphi} = R + \mathbf{i}\omega L - \mathbf{i}\frac{1}{\omega C}$$

Thus, we can express the total impedance $Z \equiv (V_0 / I_0)\, e^{\mathbf{i}\varphi}$ of the circuit as a complex number with the magnitude $|Z| = V_0 / I_0$ and phase φ.

In this representation the equivalent resistances (reactances) of L and C are imaginary numbers

$$X_L = \mathbf{i}\omega L \quad \text{and} \quad X_C = -\frac{\mathbf{i}}{\omega C}$$

Plotted in the complex plane:



Relative to V_R, the voltage across the inductor V_L is shifted in phase by +90°.

The voltage across the capacitor V_C is shifted in phase by -90°.

The total impedance has the magnitude

$$|Z| = \sqrt{[\mathrm{Re}(Z)]^2 + [\mathrm{Im}(Z)]^2} = \sqrt{R^2 + \left(wL - \frac{1}{wC}\right)^2}$$

and the phase $$\tan j = \frac{\mathrm{Im}(Z)}{\mathrm{Re}(Z)} = \frac{wL - \frac{1}{wC}}{R}$$

From this one sees immediately that the impedance Z assumes a minimum at

$$w = \frac{1}{\sqrt{LC}} \ ,$$

the resonant frequency of the tuned circuit. The impedance vs. frequency yields the resonance curve. At resonance the phase φ becomes zero.

At frequencies above resonance the inductive reactance dominates (as in the drawing above) and the asymptotic phase is +90°.

Below resonance the capacitive reactance dominates and the asymptotic phase
is -90°.

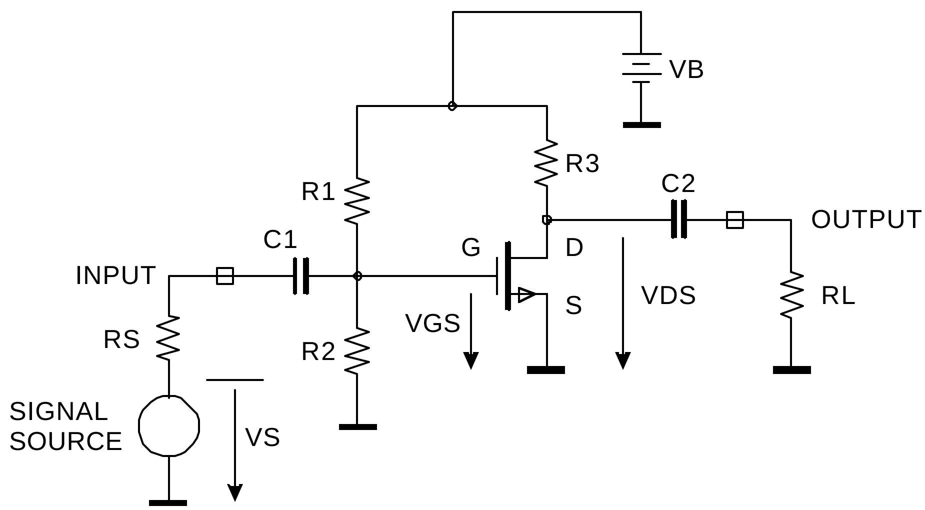
Use to represent any element that introduces a phase shift, e.g an amplifier. A phase shift of +90° appears as $+\mathbf{i}$, -90° as $-\mathbf{i}$.

Appendix 2

Equivalent Circuits

Take a simple amplifier as an example.

a) full circuit diagram



First, just consider the DC operating point of the circuitry between *C1* and *C2*:

1. The n-type MOSFET requires a positive voltage applied from the gate G to the source S.

$$V_{GS} = \frac{R2}{R1+R2} V_B$$

2. The gate voltage V_{GS} sets the current flowing into the drain electrode D.

3. Assume the drain current is I_D. Then the DC voltage at the drain is

$$V_{DS} = V_B - I_D R3$$

Next, consider the AC signal V_S provided by the signal source.

Assume that the signal at the gate G is dV_G/dt.

1. The current flowing through *R2* is

$$\frac{dI}{dt}(R2) = \frac{dV_G}{dt} \cdot \frac{1}{R2}$$

2. The current flowing through *R1* is

$$\frac{dI}{dt}(R1) = \frac{1}{R1} \cdot \frac{d}{dt}(V_G + \mathrm{V_B})$$

Since the battery voltage $\mathrm{V_B}$ is constant,

$$\frac{d\mathrm{V_B}}{dt} = 0$$

so that

$$\frac{dI}{dt}(R1) = \frac{1}{R1} \cdot \frac{dV_G}{dt}$$

3. The total time-dependent input current is

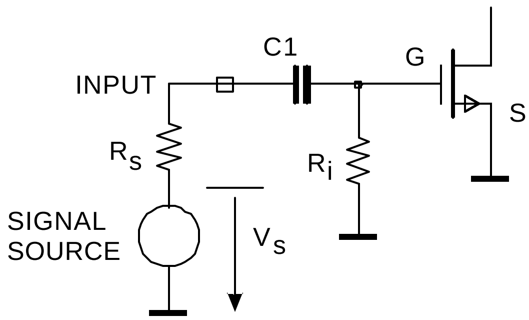
$$\frac{dI}{dt} = \frac{dI_{R1}}{dt} + \frac{dI_{R2}}{dt} = \left(\frac{1}{R1} + \frac{1}{R2}\right) \cdot \frac{dV_G}{dt} \equiv \frac{1}{R_i} \cdot \frac{dV_G}{dt}$$

where

$$R_i = \frac{R1 \cdot R2}{R1 + R2}$$

is the parallel connection of *R1* and *R2*.

Consequently, for the AC input signal the circuit is equivalent to



At the output, the voltage signal is formed by the current of the transistor flowing through the combined output load formed by R_L and $R3$.

For the moment, assume that $R_L >> R3$. Then the output load is dominated by $R3$.

The voltage at the drain D is

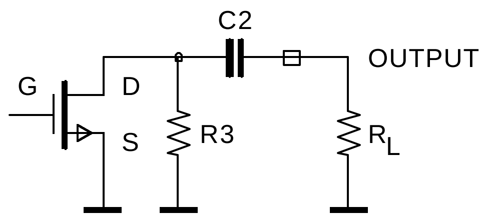
$$V_o = \mathrm{V_B} - i_D R3$$

If the gate voltage is varied, the transistor drain current changes, with a corresponding change in output voltage

$$\frac{dV_o}{di_D} = \frac{d}{dI_D}(\mathrm{V_B} - i_D R3) = R3$$

Þ The DC supply voltage does not directly affect the signal formation.

If we remove the restriction $R_L >> R3$, the total load impedance for time-variant signals is the parallel connection of $R3$ and $(X_{C2} + R_L)$, yielding the equivalent circuit at the output



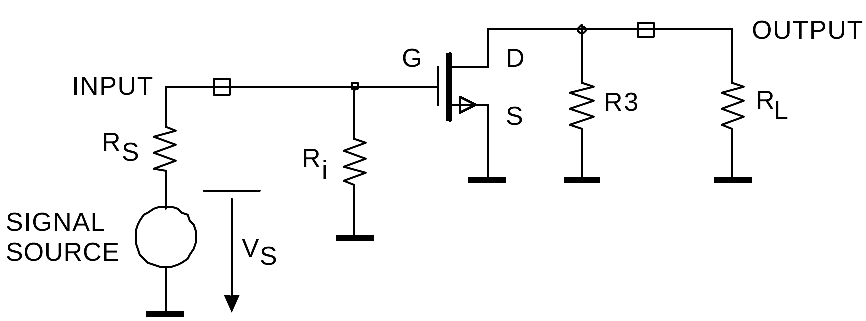
If the source resistance of the signal source $R_S << R_i$, the input coupling capacitor $C1$ and input resistance R_i form a high-pass filter. At frequencies where the capacitive reactance is $<<R_i$, i.e.

$$f >> \frac{1}{2p\ R_i C1}$$

the source signal v_s suffers negligible attenuation at the gate, so that

$$\frac{dV_G}{dt} = \frac{dV_s}{dt}$$

Correspondingly, at the output, if the impedance of the output coupling capacitor $C2 << R_L$, the signal across R_L is the same as across $R3$, yielding the simple equivalent circuit



Note that this circuit is only valid in the “high-pass” frequency regime.

Equivalent circuits are an invaluable tool in analyzing systems, as they remove extraneous components and show only the components and parameters essential for the problem at hand.

Often equivalent circuits are tailored to very specific questions and include simplifications that are not generally valid. Conversely, focussing on a specific question with a restricted model may be the only way to analyze a complicated situation.

Appendix 3: Noise Spectral Densities

Spectral Density of Thermal Noise

Two approaches can be used to derive the spectral distribution of thermal noise.

1. The thermal velocity distribution of the charge carriers is used to calculate the time dependence of the induced current, which is then transformed into the frequency domain.

2. Application of Planck's theory of black body radiation.

The first approach clearly shows the underlying physics, whereas the second "hides" the physics by applying a general result of statistical mechanics. However, the first requires some advanced concepts that go well beyond the standard curriculum, so the "black body" approach will be used.

In Planck's theory of black body radiation the energy per mode

$$\overline{E} = \frac{hn}{e^{hn/kT} - 1}$$

and the spectral density of the radiated power

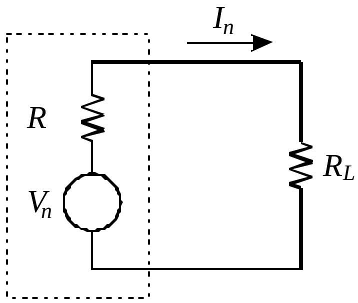
$$\frac{dP}{dn} = \frac{hn}{e^{hn/kT} - 1}$$

i.e. this is the power that can be extracted in equilibrium. At low frequencies $hn \ll kT$

$$\frac{dP}{dn} \approx \frac{hn}{\left(1 + \frac{hn}{kT}\right) - 1} = kT \ ,$$

so at low frequencies the spectral density is independent of frequency and for a total bandwidth B the noise power that can be transferred to an external device $P_n = kTB$.

To apply this result to the noise of a resistor, consider a resistor R whose thermal noise gives rise to a noise voltage V_n. To determine the power transferred to an external device consider the circuit



The power dissipated in the load resistor R_L

$$\frac{V_{nL}^2}{R_L} = I_n^2 R_L = \frac{V_n^2 R_L}{(R+R_L)^2}$$

The maximum power transfer occurs when the load resistance equals the source resistance $R_T = R$, so

$$V_{nL}^2 = \frac{V_n^2}{4}$$

Since the power transferred to R_L is kTB

$$\frac{V_{nL}^2}{R} = \frac{V_n^2}{4R} = kTB$$

$$P_n = \frac{V_n^2}{R} = 4kTB$$

and the spectral density of the noise power

$$\frac{dP_n}{dn} = 4kT$$

Spectral Density of Shot Noise

If an excess electron is injected into a device, it forms a current pulse of duration t. In a thermionic diode t is the transit time from cathode to anode (see IX.2), for example. In a semiconductor diode t is the recombination time (see IX-2). If these times are short with respect to the periods of interest $t \ll 1/f$, the current pulse can be represented by a d pulse. The Fourier transform of a delta pulse yields a "white" spectrum, i.e. the amplitude distribution in frequency is uniform

$$\frac{dI_{n,pk}}{df} = 2q_e$$

Within an infinitesimally narrow frequency band the individual spectral components are pure sinusoids, so their rms value

$$i_n \equiv \frac{dI_n}{df} = \frac{2q_e}{\sqrt{2}} = \sqrt{2}q_e$$

If N electrons are emitted at the same average rate, but at different times, they will have the same spectral distribution, but the coefficients will differ in phase. For example, for two currents i_p and i_q with a relative phase j the total rms current

$$\langle i^2 \rangle = \left(i_p + i_q e^{ij}\right)\left(i_p + i_q e^{-ij}\right) = i_p^2 + i_q^2 + 2 i_p i_q \cos j$$

For a random phase the third term averages to zero

$$\langle i^2 \rangle = i_p^2 + i_q^2 \;,$$

so if N electrons are randomly emitted per unit time, the individual spectral components simply add in quadrature

$$i_n^2 = 2Nq_e^2$$

The average current

$$I = Nq_e \;,$$

so the spectral noise density

$$i_n^2 \equiv \frac{dI_n^2}{df} = 2q_e I$$

“Noiseless” Resistances

a) Dynamic Resistance

In many instances a resistance is formed by the slope of a device’s current-voltage characteristic, rather than by a static ensemble of electrons agitated by thermal energy.

Example: forward-biased semiconductor diode

Diode current vs. voltage

$$I = I_0 (e^{q_e V / kT} - 1)$$

The differential resistance

$$r_d = \frac{dV}{dI} = \frac{kT}{q_e I}$$

i.e. at a given current the diode presents a resistance, e.g. 26 Ω at I = 1 mA and T = 300 K.

Note that two diodes can have different charge carrier concentrations, but will still exhibit the same dynamic resistance at a given current, so the dynamic resistance is not uniquely determined by the number of carriers, as in a resistor.

There is no thermal noise associated with this “dynamic” resistance, although the current flow carries shot noise.

b) Radiation Resistance of an Antenna

Consider a receiving antenna with the normalized power pattern $P_n(q,f)$ pointing at a brightness distribution $B(q,f)$ in the sky. The power per unit bandwidth received by the antenna

$$w = \frac{A_e}{2}\iint B(q,f)P_n(q,f)d\Omega$$

where A_e is the effective aperture, i.e. the "capture area" of the antenna. For a given field strength E, the captured power $W \propto EA_e$.

If the brightness distribution is from a black body radiator and we're measuring in the Rayleigh-Jeans regime,

$$B(q,f) = \frac{2kT}{l^2}$$

and the power received by the antenna

$$w = \frac{kT}{l^2} A_e \Omega_A \; .$$

Ω_A is the beam solid angle of the antenna (measured in rad^2), i.e. the angle through which all the power would flow if the antenna pattern were uniform over its beamwidth.

Since $A_e \Omega_A = l^2$ (see antenna textbooks), the received power

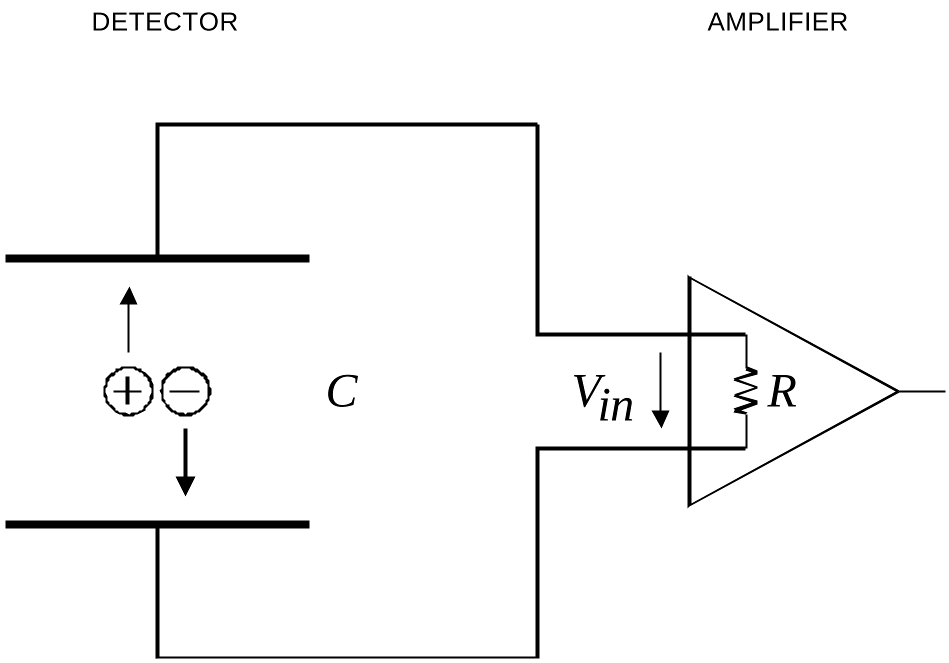
$$w = kT$$

The received power is independent of the radiation resistance, as would be expected for thermal noise.

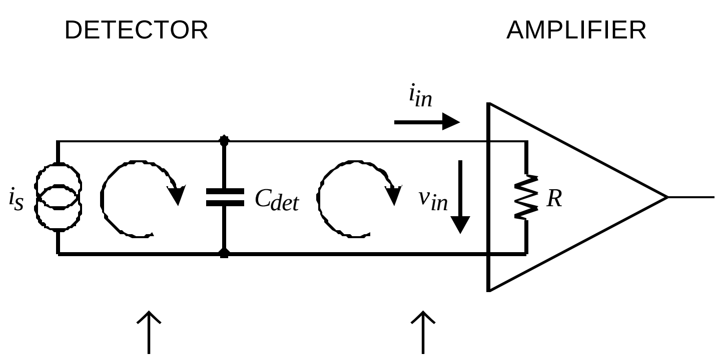
However, it is not determined by the temperature of the antenna, but by the temperature of the sky the antenna pattern is subtending.

For example, for a region dominated by the CMB, the measured power corresponds to a resistor at a temperature of ~3K, although the antenna may be at 300K.

Appendix 4
Signal-to-Noise Ratio vs. Detector Capacitance



Equivalent Circuit



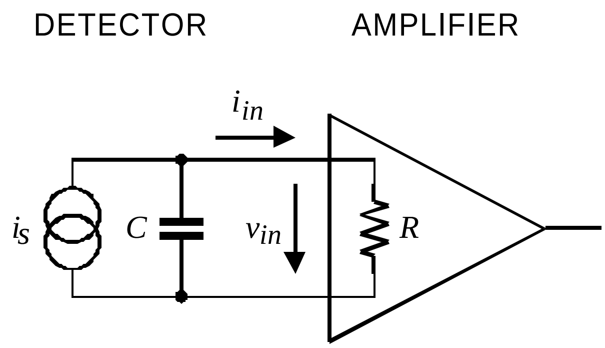
charges moving in detector induce change of charge on detector electrodes

detector capacitance discharges into amplifier

Assume an amplifier with constant noise. Then signal-to-noise ratio (and the equivalent noise charge) depend on the signal magnitude. Pulse shape registered by amplifier depends on the input time constant RC_{det}.

Assume a rectangular detector current pulse of duration T and magnitude I_s.

Equivalent circuit



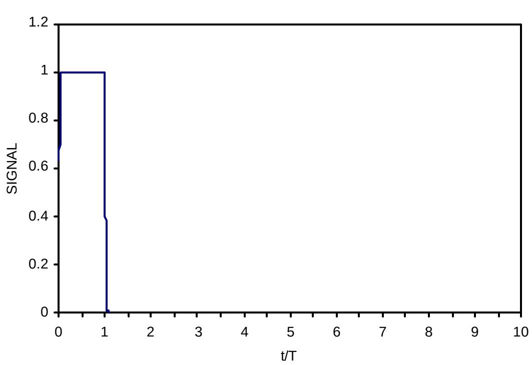
Input current to amplifier

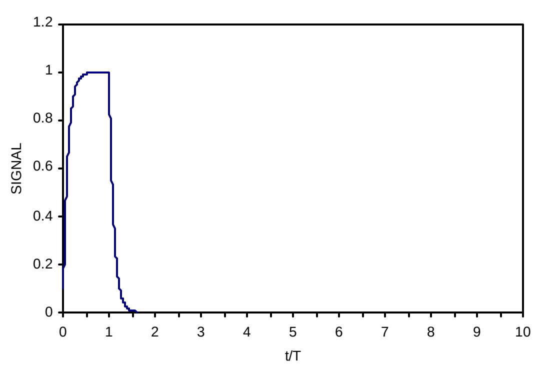
$$0 \le t < T: \qquad i_{in}(t) = I_s\left(1 - e^{-t/RC}\right)$$

$$T \le t \le \infty: \qquad i_{in}(t) = I_s\left(e^{T/RC} - 1\right) \cdot e^{-t/RC}$$

At short time constants $RC << T$ the amplifier pulse approximately follows the detector current pulse.

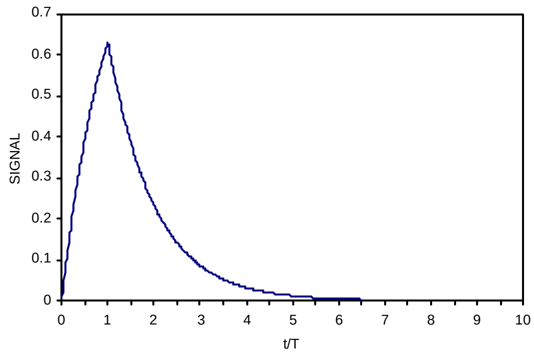
$RC = 0.01\ T$ $\qquad$ $RC = 0.1\ T$



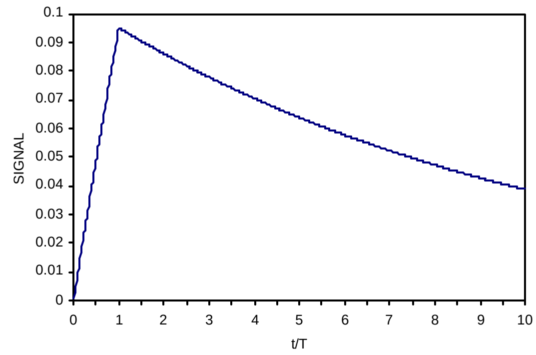


As the input time constant RC increases, the amplifier signal becomes longer and the peak amplitude decreases, although the integral, i.e. the signal charge, remains the same.

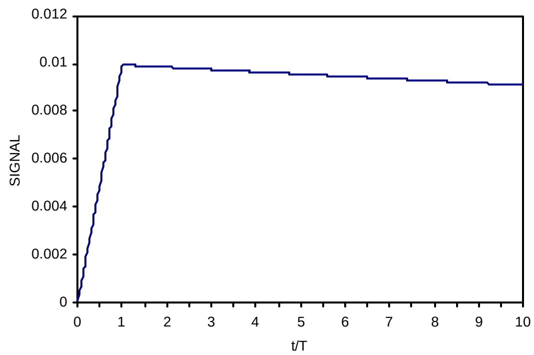
$RC = T$



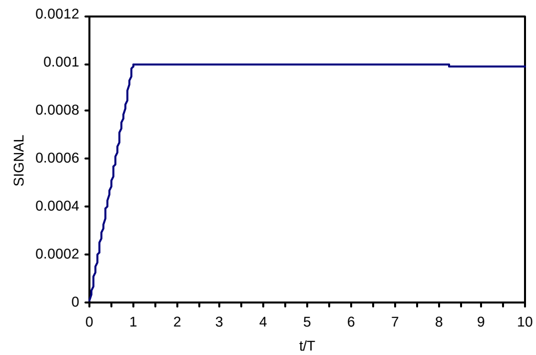
$RC = 10\ T$



$RC = 100\ T$



$RC = 10^3\ T$



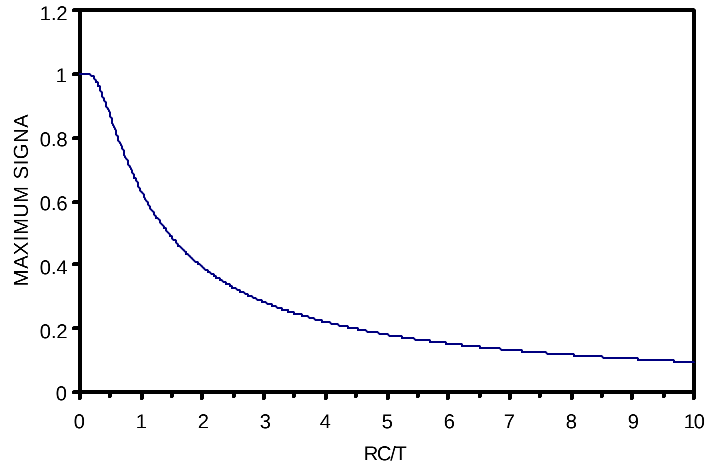
At long time constants the detector signal current is integrated on the detector capacitance and the resulting voltage sensed by the amplifier

$$V_{in} = \frac{Q_{\det}}{C} = \frac{\int i_s dt}{C}$$

Then the peak amplifier signal is inversely proportional to the **total capacitance at the input**, i.e. the sum of

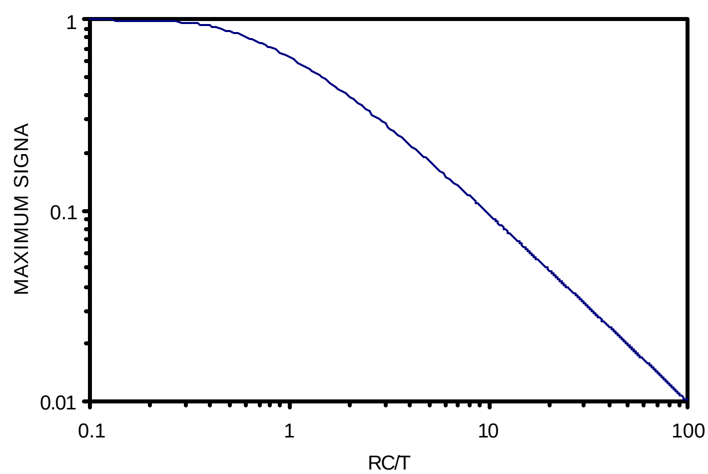
detector capacitance,
input capacitance of the amplifier, and
stray capacitances.

Maximum signal vs. capacitance



At small time constants the amplifier signal approximates the detector current pulse and is independent of capacitance.

At large input time constants (RC/T > 5) the maximum signal falls linearly with capacitance.



Þ For input time constants large compared to the detector pulse duration the signal-to-noise ratio decreases with detector capacitance.

Caution when extrapolating to smaller capacitances:

If S/N = 1 at RC/T = 100, decreasing the capacitance to 1/10 of its original value (RC/T = 10), increases S/N to 10.

However, if initially RC/T = 1, the same 10-fold reduction in capacitance (to RC/T = 0.1) only yields S/N = 1.6.

Appendix 5: Noise in Transistors

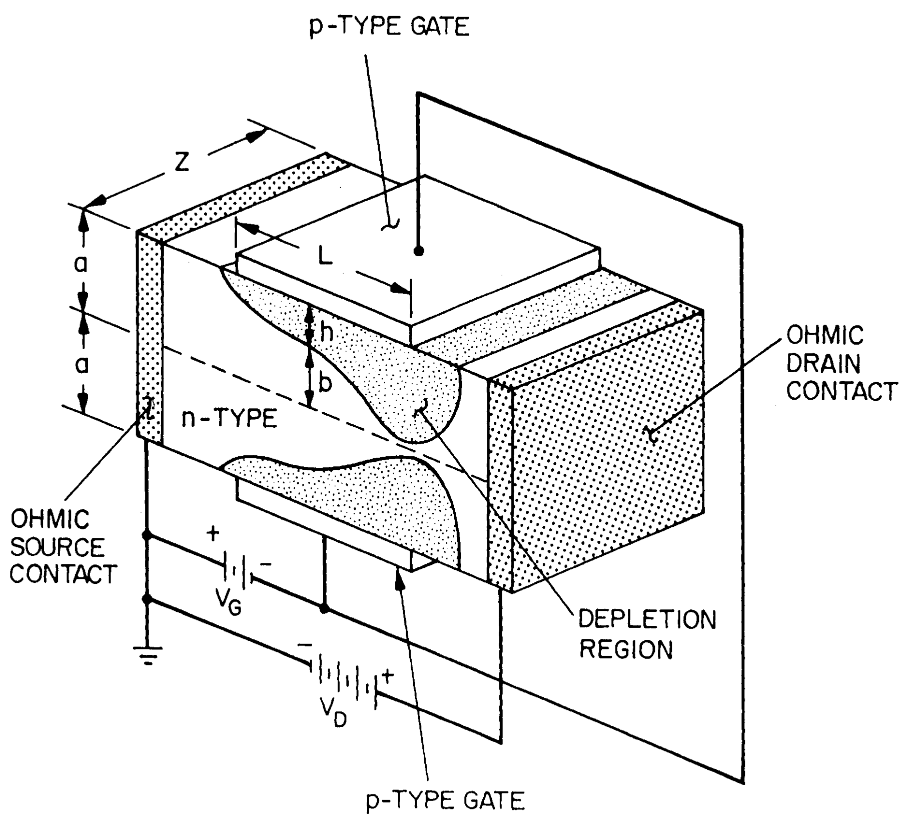
a) Field Effect Transistors

Field Effect Transistors (FETs) utilize a conductive channel whose resistance is controlled by an applied potential.

1. Junction Field Effect Transistor (JFET)

In JFETs a conducting channel is formed of *n* or *p*-type semiconductor (GaAs, Ge or Si).
Connections are made to each end of the channel, the Drain and Source.

In the implementation shown below a pair of gate electrodes of opposite doping with respect to the channel are placed at opposite sides of the channel. Applying a reverse bias forms a depletion region that reduces the cross section of the conducting channel.



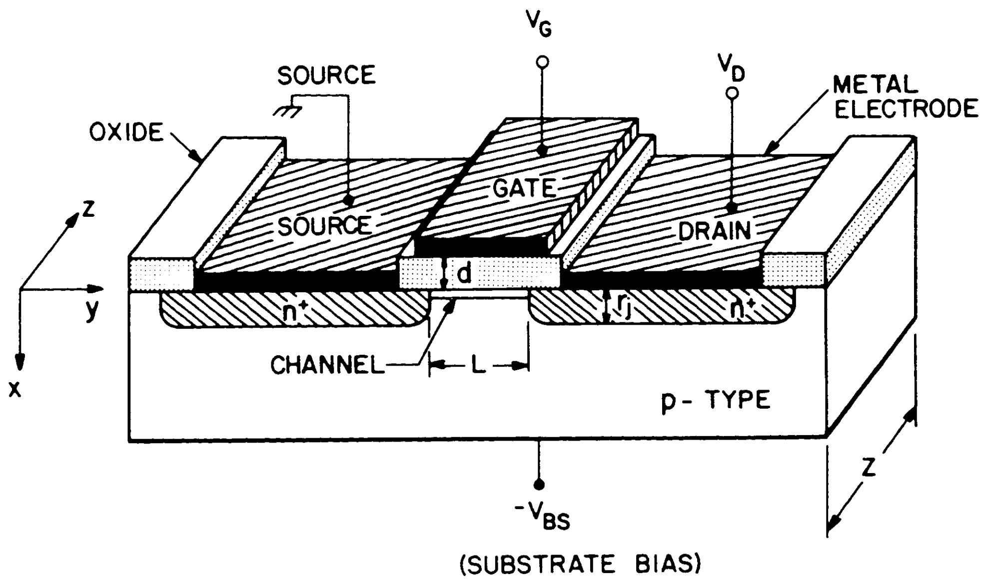
(from Sze)

Changing the magnitude of the reverse bias on the gate modulates the cross section of the channel.

2. Metal Oxide Field Effect Transistors (MOSFETs)

Both JFETs and MOSFETs are conductivity modulated devices, utilizing only one type of charge carrier. Thus they are called unipolar devices, unlike bipolar transistors, for which both electrons and holes are crucial.

Unlike a JFET, where a conducting channel is formed by doping and its geometry modulated by the applied voltages, the MOSFET changes the carrier concentration in the channel.



(from Sze)

The source and drain are n^+ regions in a p-substrate.

The gate is capacitively coupled to the channel region through an insulating layer, typically SiO_2.

Applying a positive voltage to the gate increases the electron concentration at the silicon surface beneath the gate.

- As in a JFET the combination of gate and drain voltages control the conductivity of the channel.
- Both JFETs and MOSFETs are characterized primarily by transconductance, i.e. the change in output current vs. input voltage

a) Noise in Field Effect Transistors

The primary noise sources in field effect transistors are

a) thermal noise in the channel

b) gate current in JFETs

Since the area of the gate is small, this contribution to the noise is very small and usually can be neglected.

Thermal velocity fluctuations of the charge carriers in the channel superimpose a noise current on the output current.

The spectral density of the noise current at the drain is

$$i_{nd}^2 = \frac{N_{C,tot} q_e}{L^2} m_0 \; 4k_B T_e$$

The current fluctuations depend on the number of charge carriers in the channel $N_{C,tot}$ and their thermal velocity, which in turn depends on their temperature T_e and low field mobility m_0. Finally, the induced current scales with $1/L$ because of Ramo's theorem.

To make practical use of the above expression it is necessary to express it in terms of directly measureable device parameters. Since the transconductance in the saturation region

$$g_m \propto \frac{W}{L} m N_{ch} d$$

one can express the noise current as

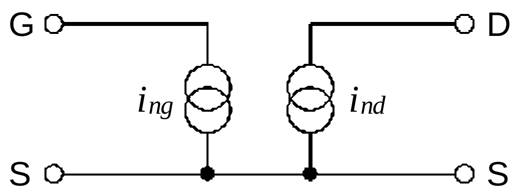
$$i_{nd}^2 = g_n g_m \; 4k_B T_0$$

where T_0= 300K and g_n is a semi-empirical constant that depends on the carrier concentration in the channel and the device geometry.

In a JFET the gate noise current is the shot noise associated with the reverse bias current of the gate-channel diode

$$i_{ng} = 2q_e I_G$$

The noise model of the FET



The gate and drain noise currents are independent of one another.

However, if an impedance Z is connected between the gate and the source, the gate noise current will flow through this impedance and generate a voltage at the gate

$$e_{ng} = Z \; i_{ng}$$

leading to an additional noise current at the output $g_m v_{ng}$, so that the total noise current at the output becomes

$$i_{no}^2 = i_{nd}^2 + (g_m Z i_{ng})^2$$

To allow a direct comparison with the input signal this cumulative noise will be referred back to the input to yield the equivalent input noise voltage

$$e_{ni}^2 = \frac{i_{no}^2}{g_m^2} = \frac{i_{nd}^2}{g_m^2} + Z i_{ng}^2 \equiv e_n^2 + Z i_n^2$$

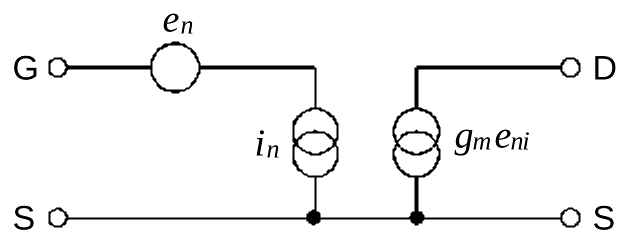
i.e. referred to the input, the drain noise current i_{nd} translates into a noise voltage source

$$e_n^2 = 4 k_B T_0 \frac{g_n}{g_m}$$

The noise coefficient g_n is usually given as 2/3, but is typically in the range 0.5 to 1 (exp. data will shown later).

This expression describes the noise of both JFETs and MOSFETs.

In this parameterization the noise model becomes



where e_n and i_n are the input voltage and current noise. As was shown above, these contribute to the input noise voltage e_{ni}, which in turn translates to the output through the transconductance g_m to yield a noise current at the output $g_m e_{ni}$.

The equivalent noise charge

$$Q_n^2 = i_n^2 F_i T + e_n^2 C_i^2 \frac{F_v}{T}$$

For a representative JFET g_m= 0.02, C_i= 10 pF and I_G < 150 pA. If F_i=F_v=1

$$Q_n^2 = 1.9 \cdot 10^9 T + \frac{3.25 \cdot 10^{-3}}{T}$$

As the shaping time T is reduced, the current noise contribution decreases and the voltage noise contribution increases. For T= 1 µs the current contribution is 43 el and the voltage contribution 3250 el, so the current contribution is negligible, except in very low frequency applications.

Optimization of Device Geometry

For a given device technology and normalized operating current I_D/W both the transconductance and the input capacitance are proportional to device width W

$$g_m \propto W \quad \text{and} \quad C_i \propto W$$

so that the ratio

$$\frac{g_m}{C_i} = const$$

Then the signal-to-noise ratio can be written as

$$\left(\frac{S}{N}\right)^2 = \frac{(Q_s/C)^2}{v_n^2} = \frac{Q_s^2}{(C_{\det}+C_i)^2}\frac{g_m}{4k_BT_0\Delta f}$$

$$\left(\frac{S}{N}\right)^2 = \frac{Q_s^2}{\Delta f}\frac{1}{4k_BT_0}\left(\frac{g_m}{C_i}\right)\frac{1}{C_i\left(1+\frac{C_{\det}}{C_i}\right)^2}$$

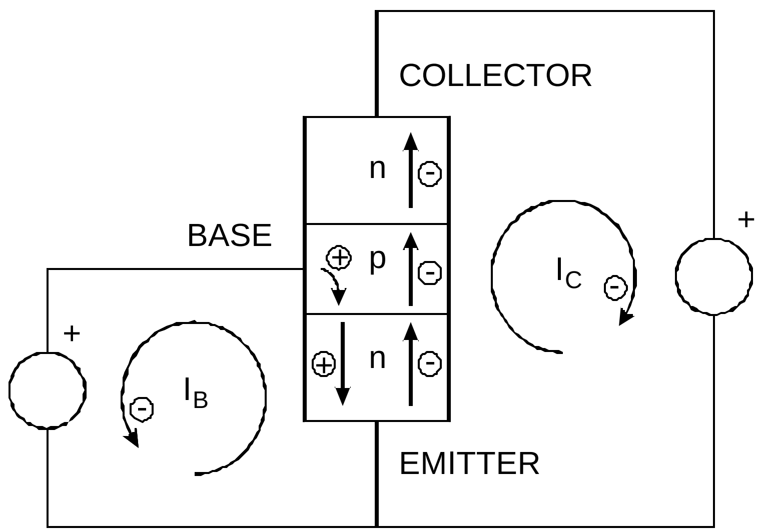
S/N is maximized for $C_i = C_{det}$ (capacitive matching).

$C_i \ll C_{det}$: The detector capacitance dominates, so the effect of increased transistor capacitance is negligible.
As the device width is increased the transconductance increases and the equivalent noise voltage decreases, so S/N improves.

$C_i > C_{det}$: The equivalent input noise voltage decreases as the device width is increased, but only with $1/\sqrt{W}$, so the increase in capacitance overrides, decreasing S/N.

Bipolar Transistors

Consider the *npn* structure shown below.



The base and emitter form a diode, which is forward biased so that a base current I_B flows.

The base current injects holes into the base-emitter junction.

As in a simple diode, this gives rise to a corresponding electron current through the base-emitter junction.

If the potential applied to the collector is sufficiently positive so that the electrons passing from the emitter to the base are driven towards the collector, an external current I_C will flow in the collector circuit.

The ratio of collector to base current is equal to the ratio of electron to hole currents traversing the base-emitter junction. In an ideal diode

$$\frac{I_C}{I_B} = \frac{I_{nBE}}{I_{pBE}} = \frac{D_n / N_A L_n}{D_p / N_D L_p} = \frac{N_D}{N_A} \frac{D_n L_p}{D_p L_n}$$

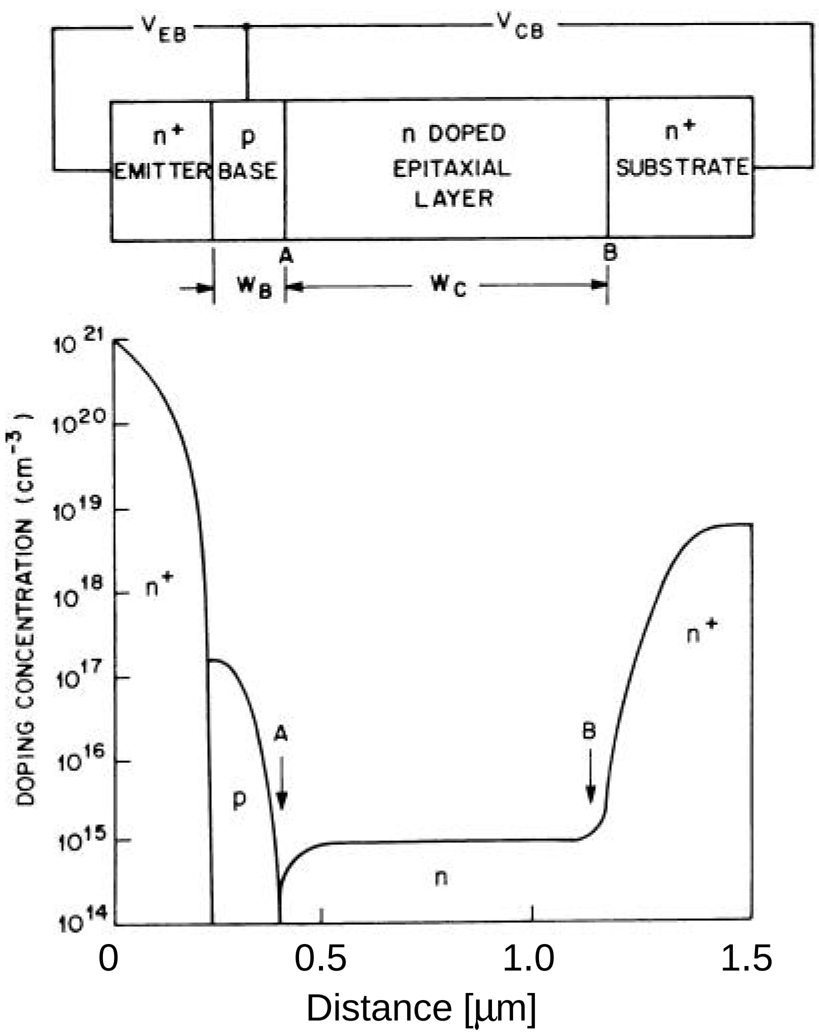
If the ratio of doping concentrations in the emitter and base regions N_D/N_A is sufficiently large, the collector current will be greater than the base current.

Þ DC current gain

Furthermore, we expect the collector current to saturate when the collector voltage becomes large enough to capture all of the minority carrier electrons injected into the base.

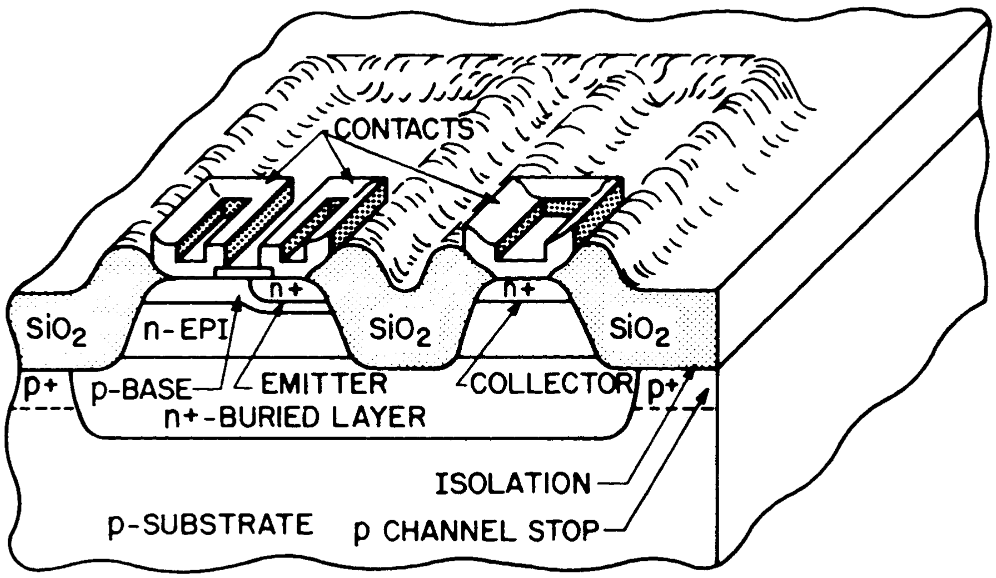
Since the current inside the transistor comprises both electrons and holes, the device is called a bipolar transistor.

Dimensions and doping levels of a modern high-frequency transistor (5 – 10 GHz bandwidth)



(adapted from Sze)

High-speed bipolar transistors are implemented as vertical structures.



(from Sze)

The base width, typically 0.2 µm or less in modern high-speed transistors, is determined by the difference in diffusion depths of the emitter and base regions.

The thin base geometry and high doping levels make the base-emitter junction sensitive to large reverse voltages.

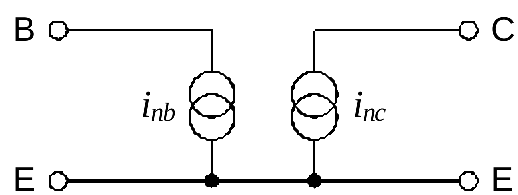
Typically, base-emitter breakdown voltages for high-frequency transistors are but a few volts.

As shown in the preceding figure, the collector region is usually implemented as two regions: one with low doping (denoted “epitaxial layer” in the figure) and the other closest to the collector contact with a high doping level. This structure improves the collector voltage breakdown characteristics.

b) Noise in Bipolar Transistors

In bipolar transistors the shot noise from the base current is important.

The basic noise model is the same as shown before, but the magnitude of the input noise current is much greater, as the base current will be 1 – 100 μA rather than <100 pA.



The base current noise is shot noise associated with the component of the emitter current provided by the base.

$$i_{nb}^2 = 2q_e I_B$$

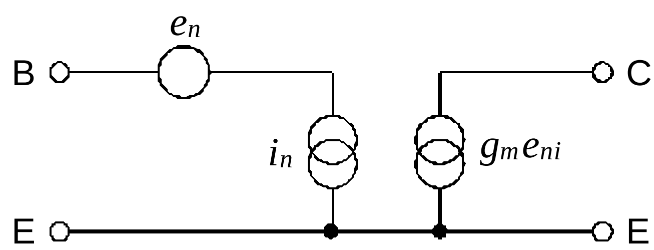
The noise current in the collector is the shot noise originating in the base-emitter junction associated with the collector component of the emitter current.

$$i_{nc}^2 = 2q_e I_C$$

Following the same argument as in the analysis of the FET, the output noise current is equivalent to an equivalent noise voltage

$$e_n^2 = \frac{i_{nc}^2}{g_m^2} = \frac{2q_e I_C}{(q_e I_C / k_B T)^2} = \frac{2(k_B T)^2}{q_e I_C}$$

yielding the noise equivalent circuit



where i_n is the base current shot noise i_{nb}.

The equivalent noise charge

$$Q_n^2 = i_n^2 F_i T + e_n^2 C^2 \frac{F_v}{T} = 2q_e I_B F_i T + \frac{2(k_B T)^2}{q_e I_C} C^2 \frac{F_v}{T}$$

Since $I_B = I_C / b_{DC}$

$$Q_n^2 = 2q_e \frac{I_C}{b_{DC}} F_i T + \frac{2(k_B T)^2}{q_e I_C} C^2 \frac{F_v}{T}$$

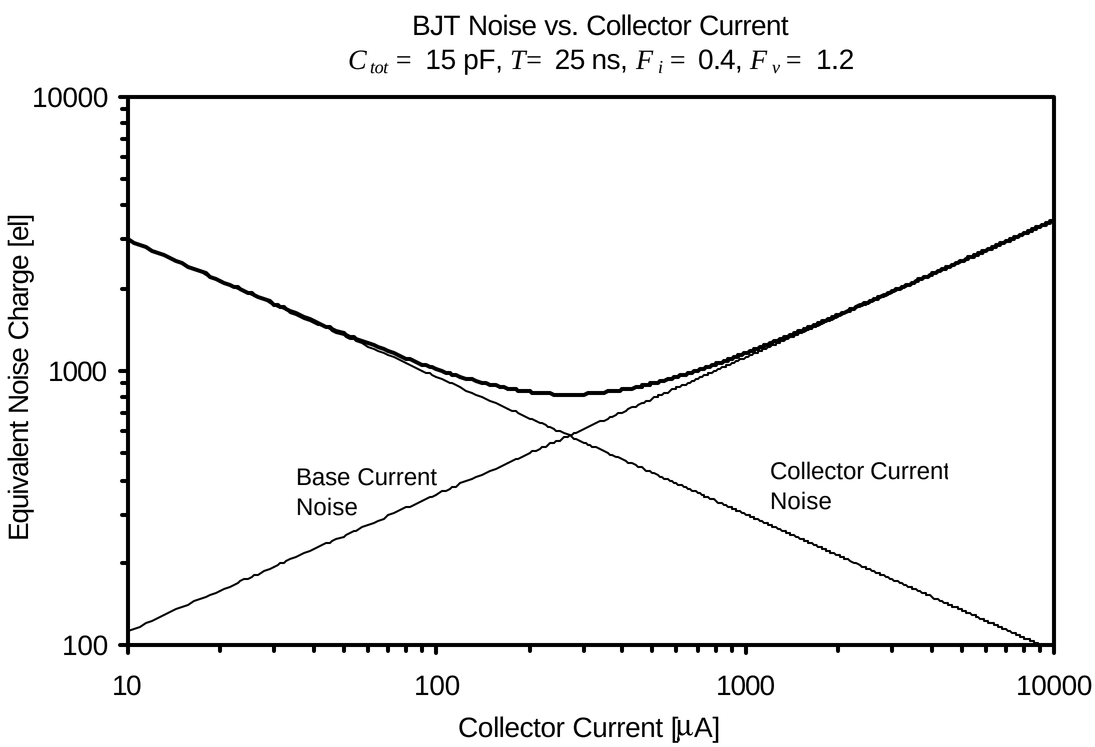
The current noise term increases with I_C, whereas the second (voltage) noise term decreases with I_C.

Thus, the noise attains a minimum

$$Q_{n,\min}^2 = 4k_B T \frac{C}{\sqrt{b_{DC}}}\sqrt{F_i F_v}$$

at a collector current

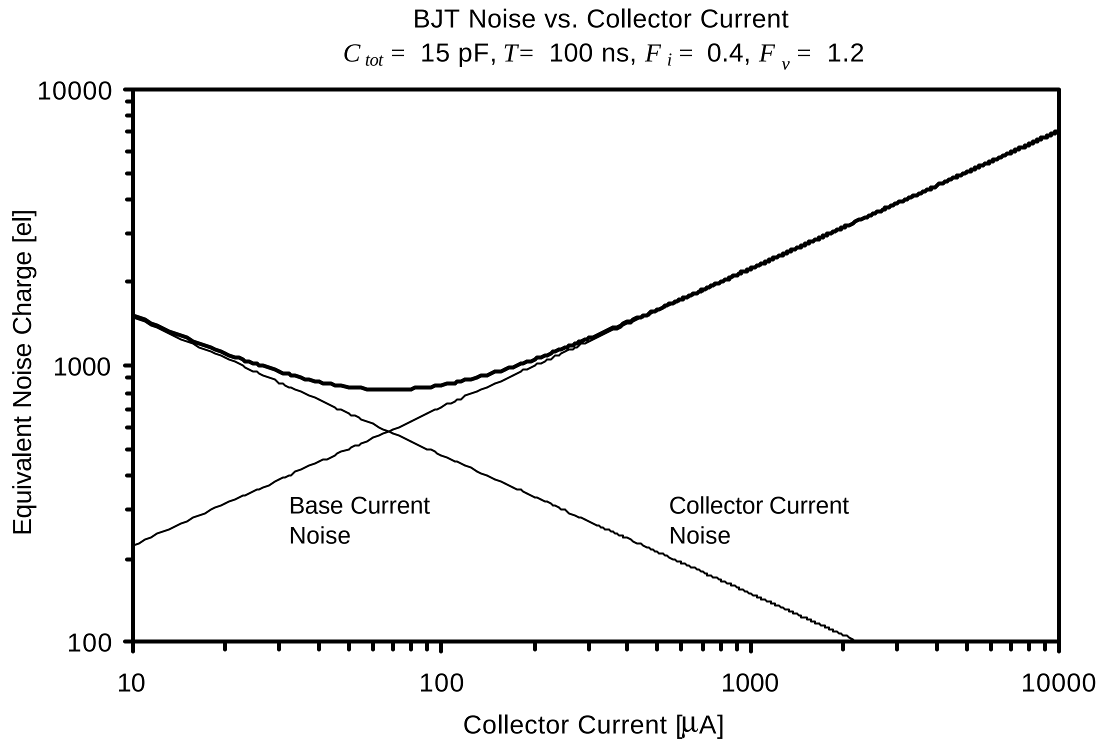
$$I_C = \frac{k_B T}{q_e} C\sqrt{b_{DC}} \sqrt{\frac{F_v}{F_i}} \frac{1}{T} .$$



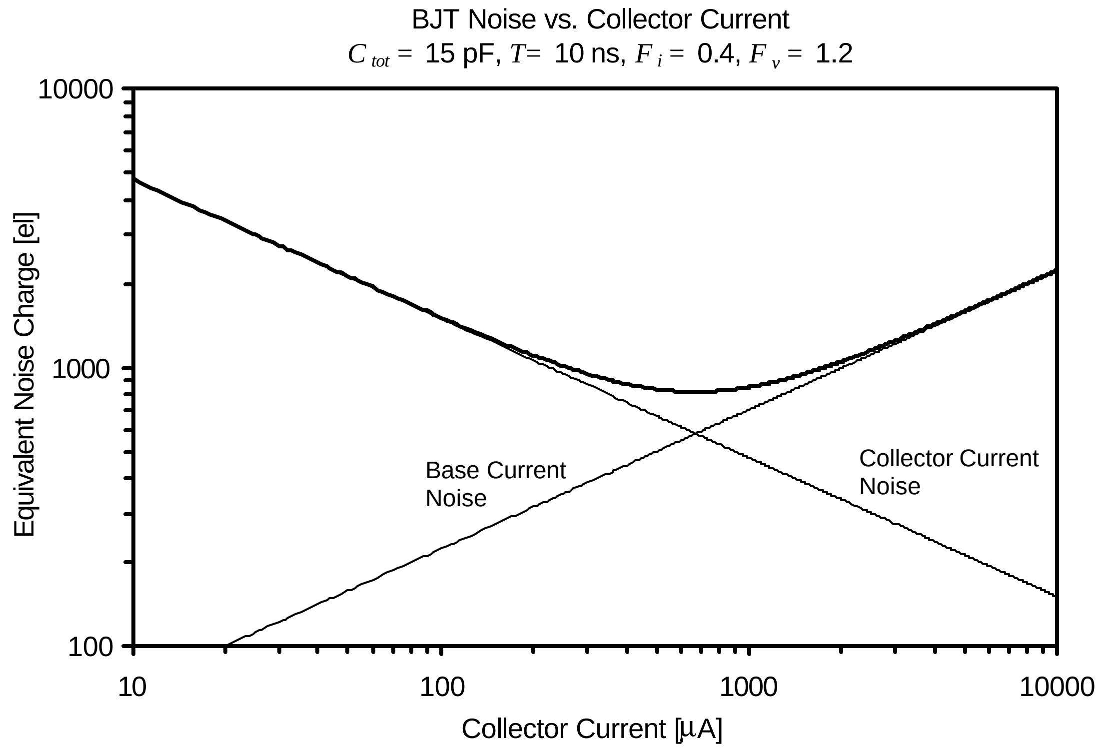
- For a given shaper, the minimum obtainable noise is determined only by the total capacitance at the input and the DC current gain of the transistor, *not by the shaping time*.

- The shaping time only determines the current at which this minimum noise is obtained

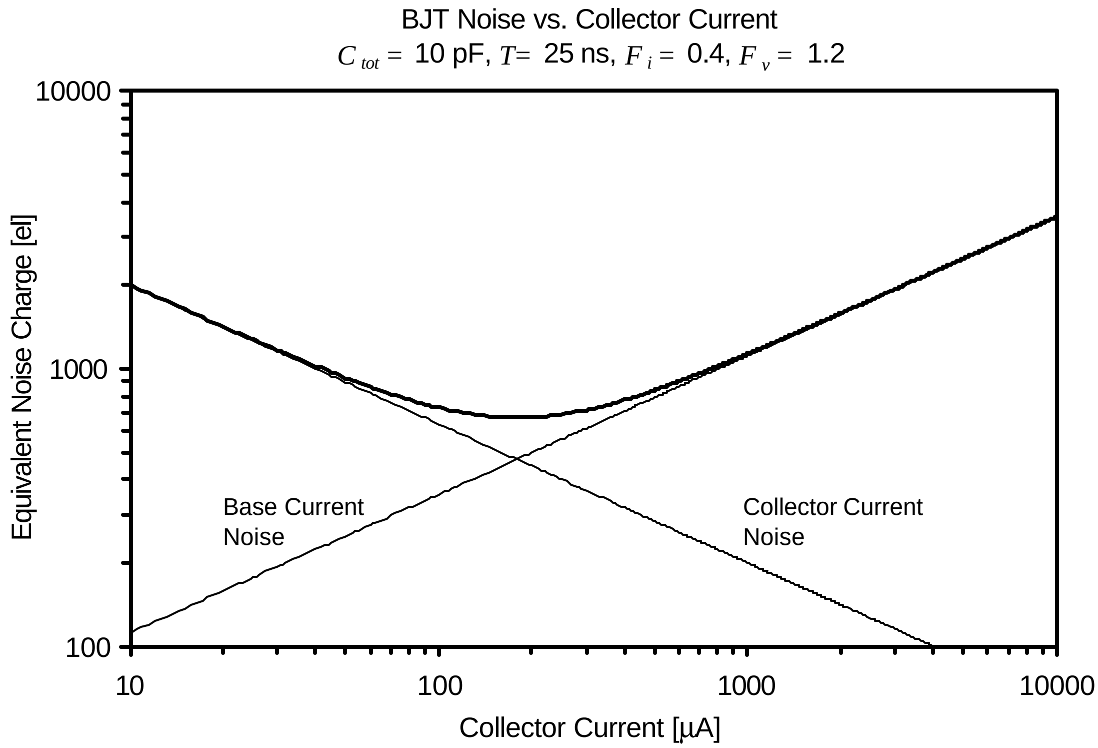
T= 100 ns

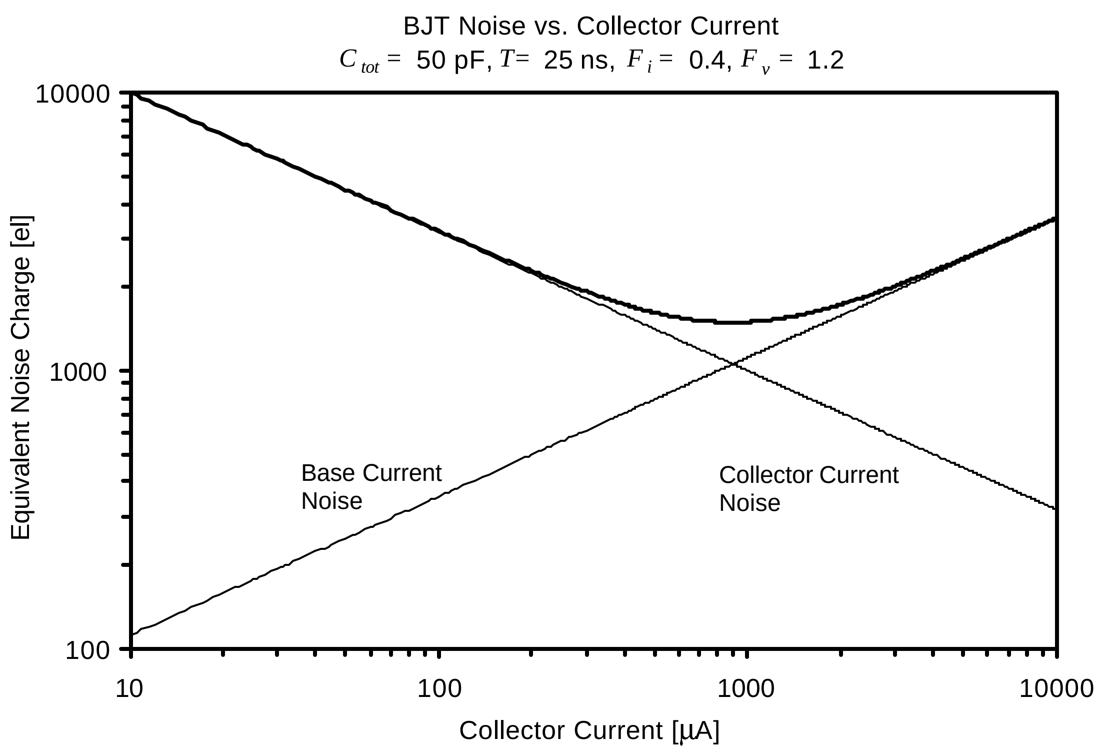


T= 10 ns



Increasing the capacitance at the input shifts the collector current noise curve upwards, so the noise increases and the minimum shifts to a higher current.





Simple Estimate of obtainable BJT noise

For a CR-RC shaper

$$Q_{n,\min} = 772 \left[\frac{el}{\sqrt{pF}} \right] \cdot \frac{\sqrt{C}}{\sqrt[4]{b_{DC}}}$$

obtained at
$$I_c = 26 \left[\frac{mA \cdot ns}{pF} \right] \cdot \frac{C}{t} \sqrt{b_{DC}}$$

Since typically $b_{DC} \approx 100$, these expressions allow a quick and simple estimate of the noise obtainable with a bipolar transistor.

Note that specific shapers can be optimized to minimize either the current or the voltage noise contribution, so both the minimum obtainable noise and the optimum current will be change with respect to the above estimates.

The noise characteristics of bipolar transistors differ from field effect transistors in four important aspects:

1. The equivalent input noise current cannot be neglected, due to base current flow.
2. The total noise does not decrease with increasing device current.
3. The minimum obtainable noise does not depend on the shaping time.
4. The input capacitance is usually negligible.

The last statement requires some explanation.

The input capacitance of a bipolar transistor is dominated by two components,

1. the geometrical junction capacitance, or transition capacitance C_{TE}, and
2. the diffusion capacitance C_{DE}.

The transition capacitance in small devices is typically about 0.5 pF.

The diffusion capacitance depends on the current flow I_E through the base-emitter junction and on the base width W, which sets the diffusion profile.

$$C_{DE} = \frac{\P q_B}{\P V_{be}} = \frac{q_e I_E}{k_B T}\left(\frac{W}{2D_B}\right) \equiv \frac{q_e I_E}{k_B T} \cdot \frac{1}{w_{Ti}}$$

where D_B is the diffusion constant in the base and w_{Ti} is a frequency that characterizes carrier transport in the base. w_{Ti} is roughly equal to the frequency where the current gain of the transistor is unity.

Inserting some typical values, I_E=100 μA and w_{Ti} =10 GHz, yields C_{DE}= 0.4 pF. The transistor input capacitance $C_{TE}+C_{DE}$= 0.9 pF, whereas FETs providing similar noise values at comparable currents have input capacitances in the range 5 – 10 pF.

Except for low capacitance detectors, the current dependent part of the BJT input capacitance is negligible, so it will be neglected in the following discussion. For practical purposes the amplifier input capacitance can be considered constant at 1 ... 1.5 pF.

This leads to another important conclusion.

Since the primary noise parameters do not depend on device size and there is no significant linkage between noise parameters and input capacitance

- Capacitive matching does not apply to bipolar transistors.

Indeed, capacitive matching is a misguided concept for bipolar transistors. Consider two transistors with the same DC current gain but different input capacitances. Since the minimum obtainable noise

$$Q_{n,\min}^2 = 4k_B T \frac{C}{\sqrt{b_{DC}}} \sqrt{F_i F_v} \;,$$

increasing the transistor input capacitance merely increases the total input capacitance C_{tot} and the obtainable noise.

When to use FETs and when to use BJTs?

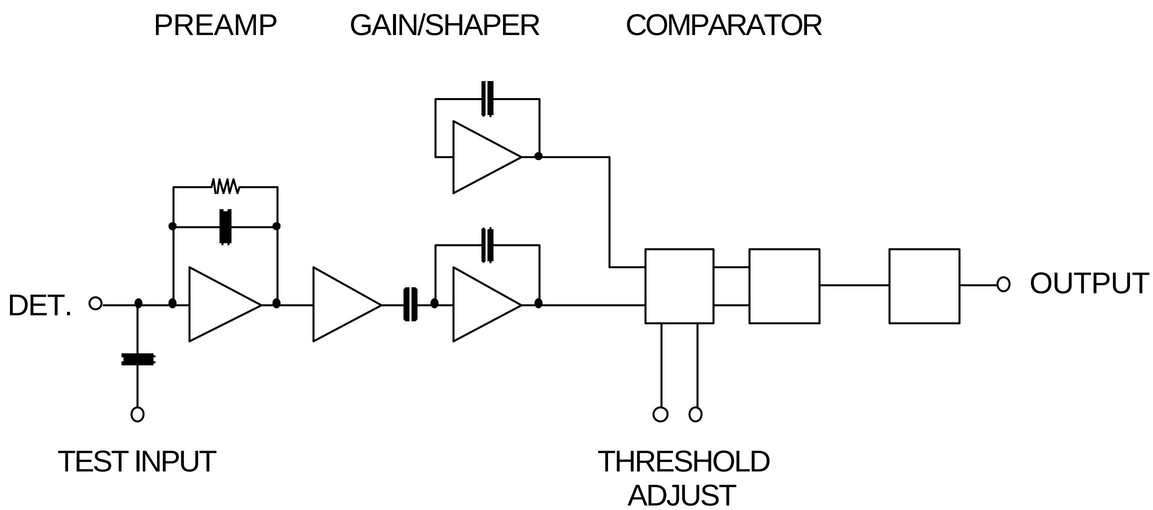
Since the base current noise increases with shaping time, bipolar transistors are only advantageous at short shaping times.

With current technologies FETs are best at shaping times greater than 50 to 100 ns, but decreasing feature size of MOSFETs will improve their performance.

Appendix 6
Rate of Noise Pulses in Threshold Discriminator Systems

Noise affects not only the resolution of amplitude measurements, but also the determines the minimum detectable signal threshold.

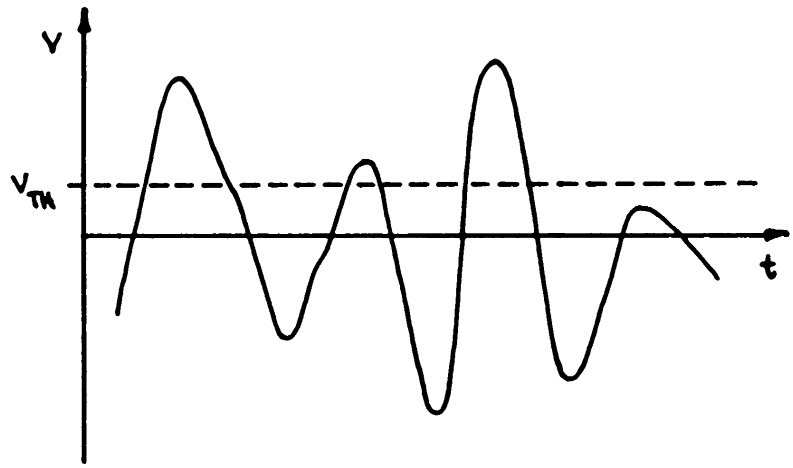
Consider a system that only records the presence of a signal if it exceeds a fixed threshold.



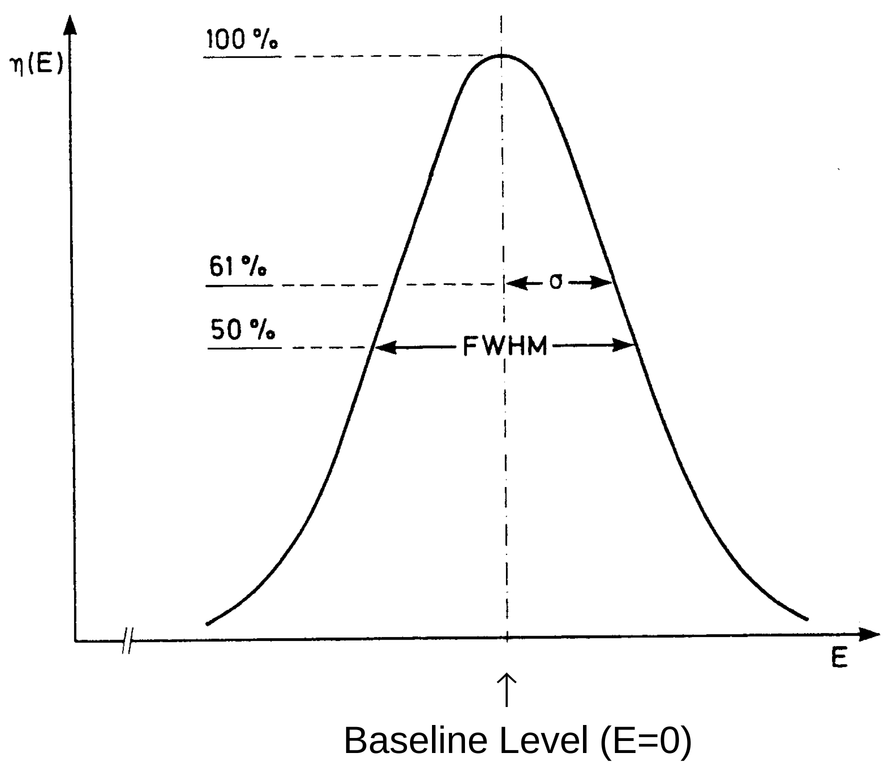
How small a detector pulse can still be detected reliably?

Consider the system at times when no detector signal is present.

Noise will be superimposed on the baseline.



The amplitude distribution of the noise is gaussian.



With the threshold level set to 0 relative to the baseline, all of the positive excursions will be recorded.

Assume that the desired signals are occurring at a certain rate.

If the detection reliability is to be >99%, then the rate of noise hits must be less than 1% of the signal rate.

The rate of noise hits can be reduced by increasing the threshold.

If the system were sensitive to pulse magnitude alone, the integral over the gaussian distribution (the error function) would determine the factor by which the noise rate f_{n0} is reduced.

$$\frac{f_n}{f_{n0}} = \frac{1}{Q_n\sqrt{2p}} \int_{Q_T}^{\infty} e^{-(Q/2Q_n)^2}\, dQ$$

where Q is the equivalent signal charge, Q_n the equivalent noise charge and Q_T the threshold level. However, since the pulse shaper broadens each noise impulse, the time dependence is equally important. For example, after a noise pulse has crossed the threshold, a subsequent pulse will not be recorded if it occurs before the trailing edge of the first pulse has dropped below threshold.

The **combined probability function** for gaussian time and amplitude distributions yields the expression for the noise rate as a function of threshold-to-noise ratio.

$$f_n = f_{n0} \cdot e^{-Q_T^2/2Q_n^2}$$

Of course, one can just as well use the corresponding voltage levels.

What is the noise rate at zero threshold f_{n0}?

Since we are interested in the number of positive excursions exceeding the threshold, f_{n0} is ½ the frequency of zero-crossings.

A rather lengthy analysis of the time dependence shows that the frequency of zero crossings at the output of an ideal band-pass filter with lower and upper cutoff frequencies f_1 and f_2 is

$$f_0 = 2\sqrt{\frac{1}{3}\,\frac{f_2^3 - f_1^3}{f_2 - f_1}}$$

(Rice, Bell System Technical Journal, **23** (1944) 282 and **24** (1945) 46)

For a CR-RC filter with $t_i = t_d$ the ratio of cutoff frequencies of the noise bandwidth is

$$\frac{f_2}{f_1} = 4.5$$

so to a good approximation one can neglect the lower cutoff frequency and treat the shaper as a low-pass filter, *i.e.* $f_1 = 0$. Then

$$f_0 = \frac{2}{\sqrt{3}}\, f_2$$

An ideal bandpass filter has infinitely steep slopes, so the upper cutoff frequency f_2 must be replaced by the noise bandwidth.

The noise bandwidth of an RC low-pass filter with time constant t is

$$\Delta f_n = \frac{1}{4t}$$

Setting $f_2 = Df_n$ yields the frequency of zeros

$$f_0 = \frac{1}{2\sqrt{3}\,t}$$

and the frequency of noise hits vs. threshold

$$f_n = f_{n0} \cdot e^{-Q_{th}^2/2Q_n^2} = \frac{f_0}{2} \cdot e^{-Q_{th}^2/2Q_n^2} = \frac{1}{4\sqrt{3}\,t} \cdot e^{-Q_{th}^2/2Q_n^2}$$

Thus, the required threshold-to-noise ratio for a given frequency of noise hits f_n is

$$\frac{Q_T}{Q_n} = \sqrt{-2\log(4\sqrt{3}\; f_n t)}$$

Note that the threshold-to-noise ratio determines the product of noise rate and shaping time, i.e. for a given threshold-to-noise ratio the noise rate is higher at short shaping times

Þ The noise rate for a given threshold-to-noise ratio is proportional to bandwidth.

Þ To obtain the same noise rate, a fast system requires a larger threshold-to-noise ratio than a slow system with the same noise level.

Frequently a threshold discriminator system is used in conjunction with other detectors that provide additional information, for example the time of a desired event.

In a collider detector the time of beam crossings is known, so the output of the discriminator is sampled at specific times.

The number of recorded noise hits then depends on

1. the sampling frequency (e.g. bunch crossing frequency) f_S

2. the width of the sampling interval Dt, which is determined by the time resolution of the system.

The product $f_S \, Dt$ determines the fraction of time the system is open to recording noise hits, so the rate of recorded noise hits is $f_S \, Dt \, f_n$.

Often it is more interesting to know the probability of finding a noise hit in a given interval, i.e. the occupancy of noise hits, which can be compared to the occupancy of signal hits in the same interval.

This is the situation in a storage pipeline, where a specific time interval is read out after a certain delay time (e.g. trigger latency)

The occupancy of noise hits in a time interval Dt

$$P_n = \Delta t \cdot f_n = \frac{\Delta t}{2\sqrt{3}\, t} \cdot e^{-Q_T^2/2Q_n^2}$$

i.e. the occupancy falls exponentially with the square of the threshold-to-noise ratio.

The dependence of occupancy on threshold can be used to measure the noise level.

$$\log P_n = \log\left(\frac{\Delta t}{2\sqrt{3}\,t}\right) - \frac{1}{2}\left(\frac{Q_T}{Q_n}\right)^2$$

so the *slope* of log P_n vs. Q_T^2 yields the noise level, *independently of the details of the shaper*, which affect only the offset.