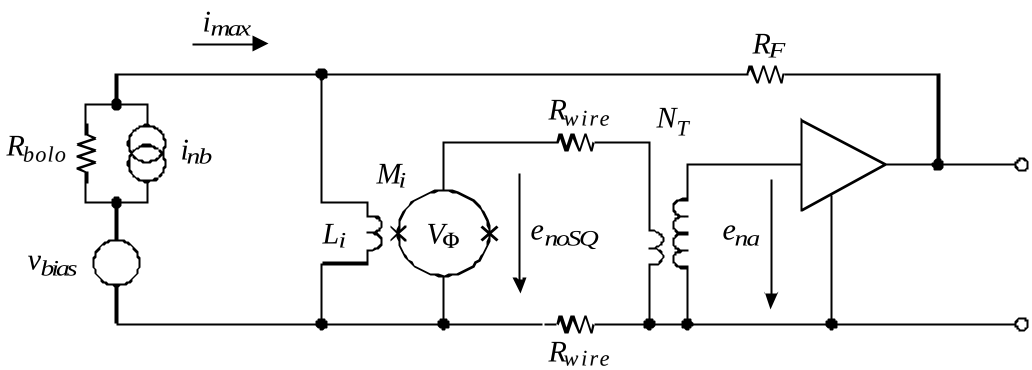
Limits on SQUID Feedback Amplifier Parameters

The purpose of this note is to develop a process for the selection of design parameters in a SQUID amplifier. The system comprises a current biased SQUID, characterized by the voltage sensitivity $V_\Phi \equiv dv_{out}/d\Phi$ and the input mutual inductance M_i. The SQUID output feeds an amplifier with a gain-bandwidth product f_0 and an equivalent input noise spectral density e_{na}. A single-pole amplifier response is assumed with an upper cutoff frequency f_{UA}. Feedback is applied from the amplifier output to the SQUID input coil through a feedback resistor R_F. The phase margin at the feedback loop's unity gain frequency is postulated to be >45°. The maximum input current is i_{max}; in the case of a TES this is dominated by the bias current of the bolometer. The bolometer noise current is i_{nb}. The goal is to accommodate the desired input signal current i_{max} over the required frequency range without significantly degrading the overall noise level beyond the sensor noise i_{nb}.



A schematic diagram of the system is shown above. Since the systems of interest here tend to drive the SQUID's input mutual inductance to values below the optimum for noise matching, the output noise voltage of the SQUID e_{noSQ} adequately characterizes the SQUID noise. A transformer with a step-up ratio N_T is inserted between the SQUID the amplifier input to improve the noise matching. To facilitate mass production, the transformer is warm, which allows the use of readily available commercial products. The resistance of the wires connecting the SQUID to the warm electronics contributes additional Johnson noise $4kTR_{wire}T_{wire}$. Since the wire connections span temperatures from 4 K to 300 K, R_{wire} and T_{wire} are equivalent values that account for the temperature dependence over length. The amplifier following the SQUID has an equivalent input noise voltage and current e_{na} and i_{na}.

The results apply to both current and voltage summing multiplexers (Section 8) and can be extended easily to a series feedback SQUID amplifier. For those who don't care to read the whole document, the key results are summarized in Section 5.

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1. Lower bound on the SQUID's Transresistance $M_i V_\Phi$

A lower bound on the SQUID's input mutual inductance and voltage sensitivity is set by the requirement that the bolometer noise override the readout noise. At the output of the SQUID the bolometer noise current generates a voltage $i_{nb} M_i V_\Phi$, which must override the total equivalent noise voltage due to the readout, referred to the SQUID output.

$$i_{nb} M_i V_\Phi \gg \sqrt{\left(\frac{e_{na}}{N_T}\right)^2 + \left(i_{na} N_T R\right)^2 + e_{noSQ}^2 + 4kT_{wire} R_{wire}} \; . \qquad (1)$$

R is the total resistance of the primary circuit, e.g. the sum of the SQUID output resistance and the wiring resistance.

If the amplifier noise voltage contribution dominates,

$$i_{nb} M_i V_\Phi \gg \frac{e_{na}}{N_T} \, . \qquad (2)$$

This is equivalent to the statement that the equivalent input noise current of the SQUID amplifier is small compared to the sensor noise current. This approximation only holds for small *N*, such that $i_{na} N_T^2 R \ll e_{na}$. In this case the amplifier is not noise matched, which may be imposed by practical constraints of the transformer.

For a given SQUID loop, i.e. a given V_Φ, this condition sets a lower bound on the SQUID's input mutual inductance M_i.

2. Lower bound on the feedback resistance R_F

A lower bound for the feedback resistance R_F is set by the requirement that its noise current be negligible compared to the spectral density of the sensor's shot noise current i_{nb},

$$\frac{4kT}{R_F} \ll i_{nb}^2 \, . \qquad (3)$$

Thus,

$$R_F \gg \frac{4kT}{i_{nb}^2} \; . \qquad (4)$$

3. Upper bound on the SQUID's input mutual inductance

3.1. Stability of the feedback loop

3.1.1. Wire lengths

The maximum usable loop gain is constrained by the propagation delay Δt due to the length of the feedback loop

$$f_{max} A_L < \frac{1}{8\Delta t} \ , \qquad (5)$$

where f_{max} is the maximum operating frequency and the feedback loop gain

$$A_L = V_\Phi N_T A_{VA} \frac{M_i}{R_F} \ . \qquad (6)$$

For stability against self-oscillation the small signal loop gain is relevant, in contrast to the maximum acceptable input signal calculated in Appendix 1, which is limited by the large signal loop gain. The amplifier gain at the maximum operating frequency is $A_{VA} = f_0 / f_{\max}$. Thus, the condition for loop stability

$$f_{max} V_\Phi N_T A_{VA} \frac{M_i}{R_F} = f_0 \frac{N_T V_\Phi M_i}{R_F} < \frac{1}{8\Delta t} \ , \qquad (7)$$

Rewriting this expression yields the maximum allowable amplifier gain-bandwidth product

$$f_0 < \frac{1}{8\Delta t} \frac{R_F}{N_T V_\Phi M_i} \ . \qquad (8)$$

As derived in Appendix 1, the product of the maximum input current at a given operating frequency is bounded (eq. A1.9)

$$f_{max} i_{max} < \frac{1}{2p} \frac{f_0 N_T V_\Phi \Phi_0}{R_F} = \frac{N_T V_P f_0}{R_F} \ . \qquad (9)$$

Combining (8) and (9) yields the maximum input current

$$i_{max} < \frac{1}{16p} \frac{\Phi_0}{N_T M_i \Delta t f_{max}} \ . \qquad (10)$$

Conversely, for a given maximum input current i_{max} this result sets an upper bound on the mutual inductance

$$M_i < \frac{1}{16p} \frac{\Phi_0}{N_T i_{max} \Delta t f_{max}} \ . \qquad (11)$$

This limit comes about because the product $f_{max} i_{max}$ sets a lower limit on $f_0 V_\Phi / R_F$. The stability criterion imposes an upper limit on the loop gain $A_L = N_T V_\Phi (f_0 / f)(M_i / R_F)$, which leads to an upper limit on M_i.

3.1.2. Feedback network

In a shunt feedback configuration, where the input coil is directly in the feedback network, the input inductance together with the feedback resistance introduces a pole at $w_{Uf} = R_F / L_i$. If its phase shift is to be negligible, the feedback pole must be well above the loop's unity gain frequency

$$\frac{R_F}{L_i} \gg 2p\, f_{max} A_L \qquad (12)$$

where $f_{\max}$ is the maximum operating frequency and A_L is the required loop gain at that frequency. At 1/10 of the pole frequency the phase shift is 6° and scales with w / w_{Uf}. For L_i = 100 nH and R_F = 1 K the pole is at 1.6 GHz, which at a loop unity gain frequency $f_{\max} A_L$ =100 MHz introduces 4° of phase shift.

3.3 Maximum rate of flux jumping

Noise transferred through the feedback loop can induce flux jumping in the SQUID. Since the SQUID bandwidth is much larger than the amplifier bandwidth, noise peaks will initiate flux jumping before feedback becomes active.

For a given input noise voltage spectral density e_{na}, low frequency gain A_{VA0}, and upper cutoff frequency f_{UA} the maximum mutual inductance (A2.7)

$$M_i < R_f \frac{\Phi_0}{4 e_n A_{VA0}} \sqrt{-\frac{1}{p f_{UA} \log\left(R_n / f_{UA}\right)}} \qquad (13)$$

The lower bound on the feedback resistance is given by (4) and (12).

4. Lower Bound on the SQUID Sensitivity V_F

4.1 Upper bound on M_i combined with minimum transresistance

The upper bound on the SQUID mutual inductance from either (11) or (13) combined with the minimum transresistance $M_i V_\Phi$ puts a lower bound on the SQUID sensitivity

$$V_\Phi > \frac{\left(M_i V_\Phi\right)_{min}}{\left(M_i\right)_{max}} \qquad (14)$$

4.2 Maximum input current

For a maximum input current i_{max} and operating frequency f_{max} the condition

$$\frac{1}{2p}\frac{f_0\,N_T\,V_\Phi\Phi_0}{R_F} > f_{max} i_{max} \tag{15}$$

applies (eq. A1.9 in Appendix 1). From this

$$V_\Phi > \frac{2p}{\Phi_0} f_{\max} i_{\max} \frac{R_F}{f_0 N_T}\,. \tag{16}$$

5. Equation Summary

The minimum SQUID transresistance is determined by the sensor noise current and the amplifier's input noise voltage

$$i_{nb} M_i V_\Phi \gg \sqrt{\left(\frac{e_{na}}{N_T}\right)^2 + \left(i_{na} N_T R\right)^2 + e_{noSQ}^2 + 4k\,T_{wire} R_{wire}}\,. \tag{17}$$

R is the total resistance of the primary circuit, e.g. the sum of the SQUID output resistance and the wiring resistance. For a small transformer turns ratio *N* the amplifier voltage noise tends to dominate and then

$$i_{nb} M_i V_\Phi \gg \frac{e_{na}}{N_T}\,. \tag{18}$$

The minimum value of the feedback resistance is set by the sensor's noise current.

$$R_F \gg \frac{4kT}{i_{nb}^2}\,. \tag{19}$$

The maximum SQUID input mutual inductance is determined by one of two criteria.

1. Stability against self-oscillation imposes a limit that depends on the maximum input current, the maximum operating frequency and the propagation delay of the feedback loop. This limit on M_i is independent of the SQUID sensitivity V_Φ and the amplifier gain.

$$M_i < \frac{1}{16p}\frac{\Phi_0}{N i_{max} \Delta t f_{max}}\,. \tag{20}$$

2. Flux jumping due to noise peaks imposes a second limit on the mutual inductance

$$M_i < R_f \frac{\Phi_0}{4e_n A_{VA0}} \sqrt{-\frac{1}{p f_{UA} \log\left(R_n / f_{UA}\right)}} \ . \tag{21}$$

The smaller of (20) and (21) together with the lower bound on $M_i V_\Phi$ (17) sets the minimum SQUID voltage sensitivity

$$V_\Phi > \frac{\left(M_i V_\Phi\right)_{min}}{\left(M_i\right)_{max}} \ . \tag{22}$$

The adopted value of V_Φ together with the minimum SQUID transresistance $M_i V_\Phi$ (17) sets the minimum input mutual inductance of the SQUID.

Eq. (17) together with the maximum allowable amplifier gain-bandwidth product

$$f_0 = \frac{1}{8\Delta t} \frac{R_F}{N_T V_\Phi M_i} \tag{23}$$

yields the allowable feedback loop gain

$$A_L = N_T V_\Phi \frac{f_0}{f_{max}} \frac{M_i}{R_F} \ . \tag{24}$$

This is to be compared with the required loop gain

$$A_{Lmin} = \frac{M_i i_{max}}{\Phi_0 / 4} - 1 \tag{25}$$

6. SQUID Series Arrays

First, we compare an array of SQUIDs with a single SQUID having the same parameters M_{i1} and $V_{\Phi 1}$ as an individual SQUID of the array. If the array has N_{SQ} SQUIDs and all are biased to provide equal response, the total output voltage for input current i_i

$$v_o = N_{SQ} i_i M_{i1} V_{\Phi 1} \ .$$

This is equivalent to introducing a step up transformer at the output of the single SQUID. The input inductance, however, is N_{SQ} times larger than for the single device.

Next, we compare a single SQUID with mutual inductance M_i and flux sensitivity V_Φ to a series array SQUID with the same total input inductance. The flux sensitivity of the individual SQUID in the array is the same as for the single SQUID. First, for simplicity assume that the SQUID noise is negligible. The output voltage for a bolometer noise current i_{nb}

$$v_{onb} = N_{SQ} M_{i1} i_{nb} V_\Phi = N_{SQ} \sqrt{\frac{L_i}{N_{SQ}} L_{SQ}}\, i_{nb} V_\Phi = \sqrt{N_{SQ} L_i L_{SQ}}\, i_{nb} V_\Phi ,$$

so for a given amplifier noise the total input inductance can be $1/N_{SQ}$ smaller, $L_{i\min} = L_i / N_{SQ}$. Then the mutual inductance of an individual SQUID in the array

$$M_{i1\min} = \sqrt{L_{i1} L_{SQ}} = \sqrt{\frac{L_{i\min}}{N_{SQ}} L_{SQ}} = \frac{1}{N_{SQ}} \sqrt{L_i L_{SQ}}$$

is $1/N$ times smaller than in the single SQUID, with a proportional decrease in deleterious capacitive coupling between the input coil and the SQUID loop.

The transresistance remains unchanged

$$\frac{dv_o}{di_i} = N_{SQ} M_{i1\min} V_\Phi = N_{SQ} \frac{1}{N_{SQ}} \sqrt{L_i L_{SQ}}\, V_\Phi = \sqrt{L_i L_{SQ}}\, V_\Phi ,$$

so the feedback loop parameters remain the same.

The smaller mutual inductance in the individual SQUID also extends the maximum input current

$$i_{max} \le \frac{\Phi_0}{4 M_{i1}} ,$$

so for the minimum input inductance

$$i_{max} \le \frac{\Phi_0}{4 M_{i1min}} = N_{SQ} \frac{\Phi_0}{4 M_i} ,$$

Thus, the maximum allowable input current also increases N_{SQ}-fold, which relaxes the requirements on the feedback loop gain. Alternatively, for the same loop gain and a given input current the intermodulation products will be reduced.

Since the input inductances of interest are smaller than required for optimum noise matching, the SQUID's input noise current dominates. As the SQUID equivalent noise current is determined primarily by the noise voltage at the SQUID output, it is convenient to express the noise in terms of the latter quantity. If the output noise voltage of a single SQUID is v_{noSQ}, then for N_{SQ} SQUIDs connected in series the output noise $\sqrt{N_{SQ}}\, v_{noSQ}$. Together with the amplifier input noise voltage v_{na}, the ratio of bolometer noise to the cumulative SQUID and amplifier noise is

$$\frac{v_{onb}}{\sqrt{v_{noSQ}^2 + v_{na}^2}} = \frac{\sqrt{N_{SQ} L_i L_{SQ}}\, i_{nb} V_\Phi}{\sqrt{N_{SQ} v_{noSQ1}^2 + v_{na}^2}},$$

The SQUID output noise voltage v_{noSQ1}, the SQUID inductance L_i and the flux sensitivity V_Φ are correlated, but since this result is predicated on a given SQUID, they are simply constants. In the limit where the amplifier noise is negligible, this ratio is independent of N_{SQ}, and for a given SQUID loop depends only the total input inductance L_i.

7. Extension to Voltage-Summing Loop

The preceding results can be applied to a voltage summing loop where each sensor is coupled to the summing loop through a transformer. The feedback signal is also coupled into the loop through a transformer. Here L_S is the secondary inductance of the sensor coupling transformer and M_S is its mutual inductance. n is the number of sensor transformers in the summing loop. The feedback transformer is characterized by its secondary inductance L_f mutual inductance M_f. As in the previous discussion, M_i is the mutual input inductance of the SQUID.

The maximum current applied to the SQUID input coil is related to the sensor current by

$$i_{max} = -\frac{n i_S}{\frac{1}{M_S}\left(nL_S + L_f + L_i\right) - \mathbf{i}\frac{\omega M_S}{R_S}}.$$

The effective mutual inductance that couples the feedback current to the SQUID loop

$$M_{feq} = -\frac{\mathbf{i}\omega M_i M_f}{\mathbf{i}\omega\left(nL_S + L_f + L_i\right) + \frac{(\omega M_S)^2}{R_S}},$$

so the loop gain

$$A_L = V_\Phi A_{VA} \frac{M_{feq}}{R_F}.$$

8. Examples

The following examples assume the following bolometer parameters: $R_S = 0.5\ \Omega$, $T_S = 0.5$ K, $G = 2 \cdot 10^{-10}$ W/K and a bias voltage $V_B = 5\ \mu$V (so $I_B = i_{max}$ = 10 μA). The total length of the feedback loop is 20 cm, so Δt = 1 ns. The inductance of the SQUID loop is 200 pH and its output noise voltage is 30 $\text{pV}/\sqrt{\text{Hz}}$. The amplifier following the SQUID has an equivalent input noise voltage of $1\ \text{nV}/\sqrt{\text{Hz}}$ and an input noise current of $2\ \text{pA}/\sqrt{\text{Hz}}$. In all cases the minimum feedback resistance is 1.5 kΩ and the total noise is 1.09 times the bolometer noise. The maximum operating frequency is 1 MHz.

8.1. No transformer between the SQUID and the next amplifier

To make the sensor noise dominate:

SQUID transresistance (eq. 18): $M_i V_\Phi = 250$ (for a 5% increase in noise)

From the maximum input current and required phase margin of feedback loop:

SQUID input mutual inductance (eq. 20): $M_i < 4.0$ nH ($L_i <$ 79 nH)

This yields the minimum SQUID sensitivity $V_\Phi > 6.2 \cdot 10^{10}$, which corresponds to a peak output voltage of 19.7 μV.

The required loop gain is 79, compared to the allowable loop gain of 125. The required amplifier gain-bandwidth product $f_0 =$ 761 MHz.

However, with the flux jumping criterion (once every 10^4 s):

SQUID input mutual inductance (eq. 21): $M_i < 15$ pH ($L_i <$ 1.1 pH)

This yields the minimum SQUID sensitivity $V_\Phi > 1.6 \cdot 10^{13}$, which corresponds to a peak output voltage of 5.2 mV.

8.2 Transformer with turns ratio 1:4 between the SQUID and the next amplifier

Assume a round-trip wiring resistance of 15 Ω and that half of this resistance is at room temperature, so that the Johnson noise spectral density $e_n^2 = 4kTR = 4k \cdot 300 \cdot 7.5$. The amplifier noise contribution is reduced through use of a warm transformer between the SQUID and the amplifier. A modest turns ratio of 4 reduces the amplifier noise voltage to

$0.25\ \mathrm{nV}/\sqrt{\mathrm{Hz}}$, compared to the wiring noise of $0.35\ \mathrm{nV}/\sqrt{\mathrm{Hz}}$. The total noise is 1.09 times the bolometer noise.

To make the sensor noise dominate:

SQUID transresistance (eq. 18): $M_i V_\Phi = 100$ (for a 5% increase in noise)

From the maximum input current and required phase margin of feedback loop:

SQUID input mutual inductance (eq. 20): $M_i < 1.0$ nH ($L_i <$ 5.0 nH)

This yields the minimum SQUID sensitivity $V_\Phi > 1.1 \cdot 10^{11}$, which corresponds to a peak output voltage of 33.4 μV.

The required loop gain is 79, compared to the allowable loop gain of 125. The required amplifier gain-bandwidth product $f_0 =$ 761 MHz.

However, with the flux jumping criterion (once every 10^4 s):

SQUID input mutual inductance (eq. 21): $M_i < 15$ pH ($L_i <$ 1.1 pH)

This yields the minimum SQUID sensitivity $V_\Phi > 7.0 \cdot 10^{12}$, which corresponds to a peak output voltage of 2.2 mV.

8.3 SQUID series array

Assume a SQUID series array with 100 SQUIDs. SQUID parameters are calculated per individual SQUID in the array.

To make the sensor noise dominate:

SQUID transresistance (eq. 18): $M_i V_\Phi = 2.5$ (for a 5% increase in noise)

From the maximum input current and required phase margin of feedback loop:

SQUID input mutual inductance (eq. 20): $M_i < 40$ pH ($L_i <$ 7.9 pH)

This yields the minimum SQUID sensitivity $V_\Phi > 6.2 \cdot 10^{10}$, which corresponds to a peak output voltage of 19.7 μV.

The required loop gain is 79, compared to the allowable loop gain of 125. The required amplifier gain-bandwidth product $f_0 =$ 761 MHz.

However, with the flux jumping criterion (once every 10^4 s):

SQUID input mutual inductance (eq. 21): $M_i < 15$ pH ($L_i < 1.1$ pH)

This yields the minimum SQUID sensitivity $V_\Phi > 1.6 \cdot 10^{11}$, which corresponds to a peak output voltage of 52 μV.

Appendix 1: Maximum Input Signal of SQUID Feedback Amplifier

The loop gain for a maximum input current i_{max} to the fedback amplifier is

$$\left|A_{LLS}\right| > \frac{i_{max}}{i_{SQmax}} - 1 \approx \frac{i_{max}}{i_{SQmax}}, \qquad \text{(A1.1)}$$

where A_{LLS} is the large signal loop gain, which at the peak of the SQUID's sinusoidal output characteristic $V_o = V_P \sin(2p\Phi/\Phi_0)$ is $2/p$ times smaller than the small signal loop gain.

The maximum current to the SQUID input $i_{SQ\,max}$ is determined by the mutual inductance M_i of the input coil to the SQUID loop

$$i_{SQ\,max} = \frac{\Phi_0/4}{M_i} \; . \qquad \text{(A1.2)}$$

The feedback loop gain is determined by the amplifier gain A_{VA}, the SQUID's sensitivity $V_\Phi \equiv dV/d\Phi$, the mutual inductance, and the feedback resistance R_F

$$A_{LLS} = \frac{2}{p} V_\Phi A_{VA} \frac{M_i}{R_F} \; . \qquad \text{(A1.3)}$$

The amplifier gain A_{VA} is required at the maximum signal frequency f_{max}, so the amplifier's gain-bandwidth product

$$f_0 = A_{VA} f_{max} \qquad \text{(A1.4)}$$

and the loop gain

$$A_{LLS} = \frac{2}{p} V_\Phi \frac{f_0}{f_{max}} \frac{M_i}{R_F} \; . \qquad \text{(A1.5)}$$

Combining this expression with the required loop gain $A_{LLS} > 4\frac{M_i i_{max}}{\Phi_0}$ from eqs. 1 and 2 yields

$$\frac{2}{p} V_\Phi \frac{f_0}{f_{max}} \frac{M_i}{R_F} > 4\frac{M_i i_{max}}{\Phi_0} \; , \qquad \text{(A1.6)}$$

so the required product of the SQUID sensitivity and the amplifier gain-bandwidth product

$$V_\Phi f_0 > 2p \frac{i_{max} f_{max} R_F}{\Phi_0} \; . \qquad \text{(A1.7)}$$

Rewritten to give a condition for $i_{max} f_{max}$ the gain criterion becomes

$$f_{\max} i_{\max} < \frac{1}{2p} \frac{f_0 V_\Phi \Phi_0}{R_F} = \frac{V_P f_0}{R_F} \quad , \tag{A1.8}$$

where the SQUID's peak output voltage $V_P = V_\Phi \Phi_0 / 2p$. Introducing a transformer with a step-up ratio N from the SQUID output to the amplifier input is equivalent to increasing V_Φ to $N V_\Phi$, so

$$f_{\max} i_{\max} < \frac{1}{2p} \frac{f_0 N V_\Phi \Phi_0}{R_F} = \frac{N V_P f_0}{R_F} \quad , \tag{A1.9}$$

The required product of the SQUID sensitivity and the amplifier gain-bandwidth product is independent of the SQUID's input mutual inductance. However, the requirement of feedback loop stability and noise considerations impose both upper and lower bounds on the mutual inductance M_i.

Appendix 2: Limits due to Flux Jumping

Noise transferred through the feedback loop can induce flux jumping in the SQUID. Since the SQUID bandwidth is much larger than the amplifier bandwidth, noise peaks will initiate flux jumping before feedback becomes active. The frequency of noise zero crossings is about equal to the upper cutoff frequency, so the time scale of the noise pulses is of order $1/f_U$. Thus, the maximum rate of change of the noise waveform is comparable to a sine wave at the amplifier cutoff frequency. To estimate the delay, consider a ramp applied to the amplifier input. At the output this is delayed by $t_d \approx 1/2p\, f_U$, so noise pulses at the amplifier output will affect the SQUID before being mitigated by negative feedback.

Assume that the amplifier has an equivalent noise voltage spectral density e_n. For an amplifier gain A_{VA} and feedback parameter M_f / R_f this introduces a fedback noise flux

$$\Phi_{nf} = e_n A_{VA} \sqrt{f_n}\, \frac{M_f}{R_f} \qquad \text{(A2.1)}$$

where f_n is the noise bandwidth of the amplifier (assuming that $R / L_f \gg 2p\, f_{UA}$). For a single-pole amplifier with an upper cutoff frequency f_{UA} the noise bandwidth

$$f_n = \frac{p}{2} f_{UA} \,. \qquad \text{(A2.2)}$$

Thus, the noise flux

$$\Phi_{nf} = e_n A_{VA0} \sqrt{\frac{p}{2} f_{UA}}\, \frac{M_f}{R_f} \;, \qquad \text{(A2.3)}$$

where A_{VA0} is the low frequency gain of the amplifier (i.e. in the constant gain regime below the cutoff frequency).

Since the noise amplitude has gaussian tails to infinite values, one needs to determine the probability of exceeding the flux limit. For a single-pole system as assumed here, the rate of noise pulses exceeding a threshold Φ_{th} [1, 2]

$$R_n = \frac{p\, f_{UA}}{2\sqrt{3}} \cdot e^{-\Phi_{th}^2 / 2\Phi_{nf}^2} \approx f_{UA} \cdot e^{-\Phi_{th}^2 / 2\Phi_{nf}^2} \,. \qquad \text{(A2.4)}$$

Here the threshold $\Phi_{th} = \Phi_0 / 4$, so

$$\log\left(\frac{R_n}{f_{UA}}\right) = -\frac{\left(\Phi_0 / 4\right)^2}{p\, f_{UA} \left(e_n A_{VA0} \dfrac{M_f}{R_f}\right)^2} \,. \tag{A2.5}$$

For a given amplifier noise, bandwidth and low frequency gain this yields the condition

$$\frac{M_f}{R_f} = \frac{\Phi_0}{4 e_n A_{VA0}} \sqrt{-\frac{1}{p\, f_{UA} \log\left(R_n / f_{UA}\right)}} \,. \tag{A2.6}$$

For a given SQUID and amplifier noise the noise rate increases exponentially with upper cutoff frequency and the square of the loop gain.

If the loop gain is decreased by reducing the gain-bandwidth product, the upper cutoff frequency remains constant and the noise rate is proportional to $\exp A_L^2$. On the other hand, if the loop gain is reduced by changing the amplifier gain, the bandwidth f_U will increase to maintain the gain-bandwidth product and the noise rate is proportional to $A_L^{-1} \exp A_L$.

Reference:

1. S.O. Rice, Bell System Technical Journal, **23** (1944) 282 and **24** (1945) 46)

2. Helmuth Spieler, course notes “Radiation Detectors and Signal Processing”, presented at University Heidelberg, Oct 2001, section IV.6.
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